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Linear perturbations of the Bloch type of space-periodic magnetohydrodynamic steady states. III. Asymptotics of branching

R. Chertovskih[1] and V. Zheligovsky*,[2]

[1]SYSTEC, ARISE, Faculty of Engineering, University of Porto, Porto, Portugal
[2]Institute of Earthquake Prediction Theory and Mathematical Geophysics RAS, Moscow, Russian Federation
* **Correspondence to:** V. Zheligovsky, vlad@mitp.ru

Abstract: The previous paper of this series presented the results of a numerical investigation of the dependence of the dominant growth rates of Bloch eigenmodes on the diffusivity parameters (the molecular viscosity ν and molecular magnetic diffusivity η) in three linear stability problems: the kinematic dynamo problem, and the hydrodynamic and MHD stability problems for steady space-periodic flows and MHD states. The dominant eigenmodes (i.e., the stability modes, whose growth rates are maximum over the wave vector $\mathbf{q}$ of the planar wave involved in the Bloch modes) comprise branches. In some branches, the dominant growth rates are attained for constant half-integer $\mathbf{q}$. In all the three stability problems for parity-invariant steady states, offshoot branches, stemming from the branches of this type, were found, in which the dominant growth rates are attained for $\mathbf{q}$ depending on ν and/or η. We consider now such a branching of the dominant magnetic modes in the kinematic dynamo problem, where an offshoot stems from a branch of neutral eigenmodes for $\mathbf{q} = 0$, and construct power series expansions for the offshoots and the associated eigenvalues of the magnetic induction operator near the point of bifurcation. We show that the branching occurs for the molecular magnetic diffusivities, for which the two eigenvalues of the eddy diffusivity operator become imaginary, and magnetic field generation by the mechanism of the negative eddy diffusivity ceases. The details of branching in the other linear stability problems under consideration are distinct.

Keywords: Power series expansion, asymptotics, kinematic dynamo problem, Bloch magnetic mode, branching, neutral eigenmodes, magnetic eddy diffusivity, scale separation.

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1. Introduction

Bloch eigenmodes of three linear stability problems: the kinematic dynamo problem, the hydrodynamic and MHD stability problem for steady space-periodic flows and MHD states were considered in the previous papers [*Chertovskih and Zheligovsky*, 2023a,b] of this series. A Bloch mode is a product of a three-dimensional vector field of the same periodicity as the perturbed state and an amplitude-modulating planar wave $e^{i\mathbf{q}\cdot\mathbf{x}}$.

We have studied the dependence of the dominant growth rates of the Bloch modes on the diffusivity parameters, the molecular magnetic diffusivity η and the molecular viscosity ν. Computations [*Chertovskih and Zheligovsky*, 2023b] have revealed that the Bloch modes, maximizing growth rates $\gamma(\mathbf{q})$ over the Bloch wave vectors $\mathbf{q}$, constitute branches, in which the dependence of the dominant growth rates on the diffusivity parameter is smooth. It was demonstrated in [*Chertovskih and Zheligovsky*, 2023a] that half-integer $\mathbf{q}$ (whose all components are integer or half-integer) satisfy the necessary condition for the maximum, $\partial\gamma/\partial q_m = 0$. In agreement with this, branches of the dominant eigenmodes were found [*Chertovskih and Zheligovsky*, 2023a], that are comprised of the eigenmodes for constant half-integer $\mathbf{q}$. Furthermore, in all the three stability problems for parity-invariant steady states, instances of branches (which we call offshoots) stemming from branches of this type (we call them main branches) were detected; the dominant growth rates in the offshoots are attained for $\mathbf{q}$ depending on ν and/or η.

The phenomenon is very common. It is observed in 9 problems out of the 18 ones which were solved in [*Chertovskih and Zheligovsky*, 2023b], see figs. 6, 9, 10, 13–15, 21–23 *ibid.* (there are two instances of branching in figs. 9, 22 and 23, and three in fig. 6). Moreover, continuation of the plots "by eye" suggests, that the branch of neutral modes for $\mathbf{q} = 0$ perhaps experiences similar branching, that we have not detected (although some of the offshoots can consist of modes that have just the locally and not globally maximum growth rates). For ν or η larger than 0.3 for which computations were performed, branching can happen in fig. 15, as well as (probably with a different asymptotics of the approach of the Bloch wave vector in the offshoots to $\mathbf{q} = 0$ in the main branch) in figs. 1, 2, 4, 6, 10, 12, 13, 18 and 20. Potential offshoots stemming from the branch of neutral modes for $\mathbf{q} = 0$ at the diffusivities below this upper bound can be observed in figs. 1 (perhaps one or two more possible branchings), 2, 12, 14, 15, 20 (the second possible branching in each of the four figures). This scenario seems impossible only for figs. 7, 16 and 19.

Consequently, questions arise: At which points (i.e., molecular diffusivities) does the branching occurs? What is the asymptotics of the offshoot near the point of bifurcation? How general is the behavior: does the asymptotics of branching coincide in the context of the three stability problems under consideration? Can offshoots stem similarly from branches of ν- and/or η-dependent eigenmodes?

In order to answer some of these questions, we consider in the present paper the problem for Bloch magnetic modes kinematically generated by a parity-invariant flow $\mathbf{V}(\mathbf{x}) = -\mathbf{V}(-\mathbf{x})$. The precise statement of this problem is outlined in the next section. We expand all the quantities involved in power series in $\vartheta = (\eta_0 - \eta)^{1/2}$, where η_0 is the magnetic diffusivity, for which branching occurs. The eigenvalue equations for the modified magnetic induction operator and its adjoint (see [*Chertovskih and Zheligovsky*, 2023a] for derivation),

$$\mathscr{D}_{\mathbf{q}} : \mathbf{b} \mapsto \eta \Delta_{\mathbf{q}} \mathbf{b} + \nabla \times (\mathbf{V} \times \mathbf{b}) + \mathrm{i}\mathbf{q} \times (\mathbf{V} \times \mathbf{b}),$$
$$\mathscr{D}^*_{\mathbf{q}} : \mathbf{b} \mapsto \eta \Delta_{\mathbf{q}} \mathbf{b} - \mathbf{V} \times (\nabla \times \mathbf{b} + \mathrm{i}\mathbf{q} \times \mathbf{b}),$$

where

$$\Delta_{\mathbf{q}} : \mathbf{f} \mapsto \nabla^2 \mathbf{f} + 2\mathrm{i}(\mathbf{q} \cdot \nabla)\mathbf{f} - |\mathbf{q}|^2 \mathbf{f}$$

is the modified Laplacian, give rise to a hierarchy of equations emerging at different orders of ϑ. In sections 3–7 we consider the equations of the hierarchy at orders ϑ^0 to ϑ^4, respectively. At order ϑ^4 we obtain the leading-order expression for the growth rate. In principle, we could also solve the equations at higher orders and obtain all terms of the expansions. The concluding remarks are summarized in the last section.

2. Statement of the problem

We consider here kinematic generation of magnetic Bloch modes by a parity-invariant flow $\mathbf{V}(\mathbf{x}) = -\mathbf{V}(-\mathbf{x})$ and investigate branching of modes featuring locally maximum growth rates from a branch of minimum-periodicity modes for $\mathbf{q} = 0$. This bifurcation is illustrated, e.g., by Fig. 14(a),(b) [*Chertovskih and Zheligovsky*, 2023b], where branch II bifurcates from branch III comprised of neutral (stationary) magnetic modes for $\mathbf{q} = 0$. We observe that at the point of bifurcation the graph of growth rates is tangent to the zero eigenvalue for $\mathbf{q} = 0$, and the locally maximum growth rates in branch II are attained for the optimal $\mathbf{q}$ that are order $\vartheta = (\eta_0 - \eta)^{1/2}$. This suggests to study the bifurcation by expanding the magnetic modes, $\mathbf{b}(\mathbf{x})$, and the associated eigenvalue, λ,

$$\mathscr{D}_{\mathbf{q}} \mathbf{b} = \lambda \mathbf{b}, \tag{1}$$

the respective eigenfunctions of the adjoint operator, $\mathbf{b}^*(\mathbf{x})$,

$$\mathscr{D}^*_\mathbf{q}\mathbf{b}^* = \overline{\lambda}\mathbf{b}^*, \tag{2}$$

and the optimal $\mathbf{q}$ in power series in ϑ:

$$\mathbf{b} = \sum_{j=0}^{\infty} \mathbf{b}_j \vartheta^j, \qquad \mathbf{b}^* = \sum_{j=0}^{\infty} \mathbf{b}^*_j \vartheta^j, \qquad \lambda = \sum_{j=1}^{\infty} \lambda_j \vartheta^j, \qquad \mathbf{q} = \sum_{j=1}^{\infty} \mathbf{q}_j \vartheta^j. \tag{3}$$

The modes are normalized by the condition

$$\langle\langle \mathbf{b}, \mathbf{b}^* \rangle\rangle = 1 \tag{4}$$

(the individual normalization of each mode is irrelevant). Here the angle brackets $\langle\langle\cdot,\cdot\rangle\rangle$ denote the scalar product

$$\langle\langle \mathbf{f}_1, \mathbf{f}_2 \rangle\rangle = \left\langle \mathbf{f}_1 \cdot \overline{\mathbf{f}_2} \right\rangle \equiv (2\pi)^{-3} \int_{\mathbb{T}^3} \mathbf{f}_1(\mathbf{x}) \cdot \overline{\mathbf{f}_2(\mathbf{x})}\, \mathrm{d}\mathbf{x}$$

in the Lebesgue space $\mathbb{L}_2(\mathbb{T}^3)$ of three-dimensional complex-valued vector fields, and the angle brackets $\langle\cdot\rangle$ denote the spatial averaging

$$\langle \mathbf{b}(\mathbf{x}) \rangle = (2\pi)^{-3} \int_{\mathbb{T}^3} \mathbf{b}(\mathbf{x})\, \mathrm{d}\mathbf{x}.$$

An eigenfunction of $\mathscr{D}_\mathbf{q}$, whose growth rate $\gamma = \mathrm{Re}\,\lambda$ is positive, automatically gives rise to a solenoidal Bloch mode. Nevertheless, it is useful to specialize the condition of solenoidality of the Bloch field $\mathrm{e}^{\mathrm{i}\mathbf{q}\cdot\mathbf{x}}\,\mathbf{b}(\mathbf{x})$ for eigenfunctions of the form of the series (3). We obtain at order ϑ^j

$$\nabla \cdot \mathbf{b}_j + \mathrm{i} \sum_{k=1}^{j} \mathbf{q}_k \cdot \mathbf{b}_{j-k} = 0.$$

In particular, $\mathbf{b}_0$ (but not $\mathbf{b}^*_0$) must be solenoidal. Averaging this equation yields

$$\sum_{k=1}^{j} \mathbf{q}_k \cdot \left\langle \mathbf{b}_{j-k} \right\rangle = 0. \tag{5}$$

We substitute the series (3) into the eigenvalue equations (1) and (2), and the conditions

$$\frac{\partial\gamma}{\partial q_m} = 0 \quad \Leftrightarrow \quad -2\eta q_m \mathrm{Re}\langle\langle \mathbf{b}, \mathbf{b}^* \rangle\rangle - \mathrm{Im}\left\langle\!\!\left\langle 2\eta \frac{\partial \mathbf{b}}{\partial x_m} + \mathbf{e}_m \times (\mathbf{V} \times \mathbf{b}), \mathbf{b}^* \right\rangle\!\!\right\rangle = 0$$

$$\Leftrightarrow \quad 2\eta\left(\mathbf{q} + \sum_m \mathrm{Im}\left\langle \frac{\partial \mathbf{b}}{\partial x_m} \cdot \overline{\mathbf{b}^*} \right\rangle \mathbf{e}_m\right) + \mathrm{Im}\left\langle (\mathbf{V} \times \mathbf{b}) \times \overline{\mathbf{b}^*} \right\rangle = 0 \tag{6}$$

for the local maximum of the growth rate γ (derived from the general expression for the gradient, see (14) in [*Chertovskih and Zheligovsky*, 2023a]).

3. Order ϑ^0 equations

Substituting $\eta = \eta_0 - \vartheta^2$ and the series (3) into (1), (2), (6) and the normalization condition (4) we obtain at order ϑ^0, respectively, the equations

$$\mathscr{D}\mathbf{b}_0 = 0, \quad (7.1)$$

$$\mathscr{D}^*\mathbf{b}_0^* = 0, \quad (7.2)$$

$$\mathrm{Im}\left\langle\!\!\left\langle 2\eta_0 \frac{\partial \mathbf{b}_0}{\partial x_m} + \mathbf{e}_m \times (\mathbf{V}\times\mathbf{b}_0), \mathbf{b}_0^* \right\rangle\!\!\right\rangle = 0, \quad (7.3)$$

$$\langle\langle \mathbf{b}_0, \mathbf{b}_0^* \rangle\rangle = 1, \quad (7.4)$$

where

$$\mathscr{D}: \mathbf{b} \mapsto \eta_0\nabla^2\mathbf{b} + \nabla\times(\mathbf{V}\times\mathbf{b}), \qquad \mathscr{D}^*: \mathbf{b} \mapsto \eta_0\nabla^2\mathbf{b} - \mathbf{V}\times(\nabla\times\mathbf{b}). \quad (8)$$

For a space-periodic flow, the magnetic induction operator $\mathscr{D}$ has at least a three-dimensional kernel of neutral modes of the same periodicity (*Arnol'd et al.* [1982], see also *Zheligovsky* [2011]). Generically, the dimension of the kernel is 3, and we assume that this holds for $\eta = \eta_0$. Then, for a parity-invariant flow $\mathbf{V}$, (7.1) implies that $\mathbf{b}_0$ is a parity-antiinvariant solenoidal field (i.e., $\mathbf{b}_0(\mathbf{x}) = \mathbf{b}_0(-\mathbf{x})$), the set of all possible $\langle\mathbf{b}_0\rangle$ spans $\mathbb{R}^3$, and

$$\mathbf{b}_0(\mathbf{x}) = \sum_{k=1}^{3} \langle\mathbf{b}_0\rangle_k \mathbf{S}_k, \quad \text{where} \quad \mathscr{D}\mathbf{S}_k = 0,\ \langle\mathbf{S}_k\rangle = \mathbf{e}_k. \quad (9)$$

Here $\langle\cdot\rangle_k$ denotes averaging of the kth component of a vector field: $\langle\mathbf{f}\rangle = \sum_{k=1}^{3}\langle\mathbf{f}\rangle_k\mathbf{e}_k$. Neutral modes $\mathbf{S}_k$ are real, solenoidal and parity-antiinvariant. From (7.2),

$$\mathbf{b}_0^* = \sum_k b_{0k}^* \mathbf{e}_k \quad (10)$$

is a constant vector. Consequently, (7.3) is satisfied identically (which agrees with the statement that half-integer $\mathbf{q}$ are stationary points of the growth rate regarded as a function of $\mathbf{q}$; it was proven in section 4 of [*Chertovskih and Zheligovsky*, 2023a]). $\{\mathbf{S}_k\}$ constitute the basis in the kernel of $\mathscr{D}$, biorthogonal to $\{\mathbf{e}_k\}$ that are the basis in the kernel of $\mathscr{D}^*$. By the Fredholm alternative theorem, the equations $\mathscr{D}\mathbf{b} = \mathbf{f}$ and $\mathscr{D}^*\mathbf{b}^* = \mathbf{f}^*$ have solutions if and only if, respectively,

$$\langle\mathbf{f}\rangle = 0, \quad (11.1)$$

$$\langle\langle \mathbf{f}^*, \mathbf{S}_k \rangle\rangle = 0 \quad \text{for all } \mathbf{S}_k \in \ker\mathscr{D}. \quad (11.2)$$

4. Order ϑ^1 equations

At order ϑ^1 we obtain from (1), (2), (6) and (4), respectively,

$$\mathscr{D}\mathbf{b}_1 + 2\mathrm{i}\eta_0(\mathbf{q}_1\cdot\nabla)\mathbf{b}_0 + \mathrm{i}\mathbf{q}_1\times(\mathbf{V}\times\mathbf{b}_0) = \lambda_1\mathbf{b}_0, \quad (12.1)$$

$$\mathscr{D}^*\mathbf{b}_1^* - \mathbf{V}\times(\mathrm{i}\mathbf{q}_1\times\mathbf{b}_0^*) = \overline{\lambda}_1\mathbf{b}_0^*, \quad (12.2)$$

$$2\eta_0\left(\mathbf{q}_1 + \sum_m \mathrm{Im}\left\langle \frac{\partial\mathbf{b}_0}{\partial x_m}\cdot\overline{\mathbf{b}_1^*}\right\rangle\mathbf{e}_m\right) + \mathrm{Im}\left\langle(\mathbf{V}\times\mathbf{b}_0)\times\overline{\mathbf{b}_1^*} + (\mathbf{V}\times\mathbf{b}_1)\times\overline{\mathbf{b}_0^*}\right\rangle = 0, \quad (12.3)$$

$$\langle\langle\mathbf{b}_0,\mathbf{b}_1^*\rangle\rangle + \langle\langle\mathbf{b}_1,\mathbf{b}_0^*\rangle\rangle = 0. \quad (12.4)$$

Averaging (12.1) yields $\lambda_1\langle\mathbf{b}_0\rangle = 0$, whereby $\lambda_1 = 0$. The conditions (11) for solvability of equations (12.1) and (12.2) are satisfied ((11.2) holds because all $\mathbf{S}_k \in \ker\mathscr{D}$ are parity-antiinvariant). The solutions are

$$\mathbf{b}_1 = \sum_k \langle\mathbf{b}_1\rangle_k\mathbf{S}_k + \mathbf{b}_{1\mathrm{p}}, \qquad \mathbf{b}_1^* = \langle\mathbf{b}_1^*\rangle + \mathbf{b}_{1\mathrm{p}}^*. \quad (13)$$

Here

$$\mathbf{b}_{1\mathrm{p}} = \mathrm{i}\sum_{k,m} q_{1m}\langle\mathbf{b}_0\rangle_k \mathbf{G}_{mk}(\mathbf{x}), \qquad \mathscr{D}\mathbf{G}_{mk} + 2\eta_0\frac{\partial\mathbf{S}_k}{\partial x_m} + \mathbf{e}_m\times(\nabla\times\mathbf{S}_k) = 0, \tag{14.1}$$

$$\mathbf{b}^*_{1\mathrm{p}} = \mathrm{i}\sum_{m,k,n}\epsilon_{mkn}\,q_{1m}\,b^*_{0k}\,\mathbf{Z}_n(\mathbf{x}), \qquad \mathscr{D}^*\mathbf{Z}_n - \mathbf{V}\times\mathbf{e}_n = 0, \tag{14.2}$$

q_{1k} are the Cartesian components of $\mathbf{q}_1 = \sum_k q_{1k}\mathbf{e}_k$ and ϵ_{mkn} is the unit antisymmetric tensor. $\mathbf{G}_{mk}(\mathbf{x})$ and $\mathbf{Z}_n(\mathbf{x})$ are real-valued parity-invariant zero-mean vector fields (since parity-invariant and parity-antiinvariant fields constitute invariant subspaces of the operators $\mathscr{D}$ and $\mathscr{D}^*$).

In the equations for $\mathbf{G}_{mk}$ and $\mathbf{Z}_n$ we recognize auxiliary problems for the magnetic induction operator (cf. section 5.2 in [*Chertovskih and Zheligovsky*, 2023a], equations (39)) and the adjoint operator (cf. section 5.3 *ibid.*) arising in the standard formalism of the analysis of magnetic eddy diffusivity in small-scale flows of electrically conducting fluid (see *Andrievsky et al.* [2015], *Rasskazov et al.* [2018], *Zheligovsky* [2011]). In fact, so far the expansion followed (except for the new condition (6) of local extremality of the growth rate) that formalism, but in the present context we have to consider the problems emerging at further levels of the asymptotic expansion.

We substitute (9), (10) and (13) into (12.3), use the identity

$$\Big\langle(\mathbf{V}\times\mathbf{S}_j)\times\mathbf{Z}_n\Big\rangle_l + 2\eta_0\left\langle\frac{\partial\mathbf{S}_j}{\partial x_l}\cdot\mathbf{Z}_n\right\rangle = \Big\langle\mathbf{V}\times\mathbf{G}_{lj}\Big\rangle_n \tag{15}$$

and obtain

$$\mathscr{E}\mathbf{q}_1 = 0, \tag{16}$$

where

$$\mathscr{E}:\mathbf{c}\mapsto 2\eta_0\,\mathbf{c} - \sum_{k,m}\Big(\big(\mathbf{c}\times\mathrm{Re}\,(\langle\mathbf{b}_0\rangle_k\overline{\mathbf{b}^*_0})\big)\cdot\langle\mathbf{V}\times\mathbf{G}_{mk}\rangle\Big)\,\mathbf{e}_m - \sum_{k,m} c_m\,\mathrm{Re}\big(\langle\mathbf{b}_0\rangle_k\overline{\mathbf{b}^*_0}\big)\times\langle\mathbf{V}\times\mathbf{G}_{mk}\rangle.$$

The linear operator $\mathscr{E}:\mathbb{R}^3\to\mathbb{R}^3$ is self-adjoint. By (16), the vector $\mathbf{q}_1$ belongs to its kernel. Generically, the kernel is one-dimensional, and then the solvability condition for the equation $\mathscr{E}\mathbf{c}=\mathbf{f}$ is the orthogonality $\mathbf{f}\cdot\mathbf{q}_1 = 0$.

5. Order ϑ^2 equations

At order ϑ^2, (1), (2), (6) and (4) give rise, respectively, to the equations

$$\mathscr{D}\mathbf{b}_2 + 2\mathrm{i}\eta_0((\mathbf{q}_1\cdot\nabla)\mathbf{b}_1 + (\mathbf{q}_2\cdot\nabla)\mathbf{b}_0) - \nabla^2\mathbf{b}_0 + \mathrm{i}\mathbf{q}_1\times(\mathbf{V}\times\mathbf{b}_1) + \mathrm{i}\mathbf{q}_2\times(\mathbf{V}\times\mathbf{b}_0)$$
$$= (\lambda_2 + \eta_0|\mathbf{q}_1|^2)\mathbf{b}_0, \tag{17.1}$$

$$\mathscr{D}^*\mathbf{b}^*_2 + 2\mathrm{i}\eta_0(\mathbf{q}_1\cdot\nabla)\mathbf{b}^*_1 - \mathrm{i}\mathbf{V}\times(\mathbf{q}_1\times\mathbf{b}^*_1 + \mathbf{q}_2\times\mathbf{b}^*_0) = (\overline{\lambda}_2 + \eta_0|\mathbf{q}_1|^2)\mathbf{b}^*_0, \tag{17.2}$$

$$2\eta_0\left(\mathbf{q}_2 + \sum_m \mathrm{Im}\left\langle\frac{\partial\mathbf{b}_0}{\partial x_m}\cdot\overline{\mathbf{b}^*_2} + \frac{\partial\mathbf{b}_1}{\partial x_m}\cdot\overline{\mathbf{b}^*_1}\right\rangle\mathbf{e}_m\right)$$
$$+\,\mathrm{Im}\left\langle(\mathbf{V}\times\mathbf{b}_0)\times\overline{\mathbf{b}^*_2} + (\mathbf{V}\times\mathbf{b}_1)\times\overline{\mathbf{b}^*_1} + (\mathbf{V}\times\mathbf{b}_2)\times\overline{\mathbf{b}^*_0}\right\rangle = 0, \tag{17.3}$$

$$\langle\langle\mathbf{b}_0,\mathbf{b}^*_2\rangle\rangle + \langle\langle\mathbf{b}_1,\mathbf{b}^*_1\rangle\rangle + \langle\langle\mathbf{b}_2,\mathbf{b}^*_0\rangle\rangle = 0. \tag{17.4}$$

The solvability condition for (17.1) is obtained by averaging (see (11.1)):

$$\mathrm{i}\,\langle\mathbf{q}_1\times(\mathbf{V}\times\mathbf{b}_1)\rangle = (\lambda_2 + \eta_0|\mathbf{q}_1|^2)\langle\mathbf{b}_0\rangle.$$

Substituting here (13) and (14.1) yields an eigenvalue equation

$$\mathfrak{D}\,\langle\mathbf{b}_0\rangle = (\lambda_2 + \eta_0|\mathbf{q}_1|^2)\langle\mathbf{b}_0\rangle \tag{18.1}$$

for the 3×3 real entry matrix $\mathfrak{D}$,

$$\mathfrak{D}\mathbf{c}=-\sum_{k,m}q_{1m}c_k\mathbf{q}_1\times\langle\mathbf{V}\times\mathbf{G}_{mk}\rangle \tag{18.2}$$

($\mathfrak{D}$ is the symbol of the operator of eddy correction of magnetic eddy diffusivity). Upon substituting (13) and (14.2), the solvability condition (11.2) for (17.2) reduces to the eigenvalue equation

$$\mathfrak{D}^*\mathbf{b}_0^*=(\overline{\lambda}_2+\eta_0|\mathbf{q}_1|^2)\mathbf{b}_0^* \tag{18.3}$$

for the 3×3 matrix $\mathfrak{D}^*$ defined by the relation

$$\mathfrak{D}^*\mathbf{c}=-\sum_{k,m,n,j}\epsilon_{mkn}q_{1m}c_k\Big\langle\Big(2\eta_0(\mathbf{q}_1\cdot\nabla)\mathbf{Z}_n-\mathbf{V}\times(\mathbf{q}_1\times\mathbf{Z}_n)\Big)\cdot\mathbf{S}_j\Big\rangle\,\mathbf{e}_j. \tag{18.4}$$

Eigenvalues of the two matrices are complex conjugate, $\mathfrak{D}^*$ being equal to the transpose of the matrix $\mathfrak{D}$ since

$$\begin{aligned}-\mathbf{q}_1\times\langle\mathbf{V}\times\mathbf{G}_{mk}\rangle&=\sum_n(\mathbf{q}_1\times\mathbf{e}_n)\langle(\mathbf{V}\times\mathbf{e}_n)\cdot\mathbf{G}_{mk}\rangle\\&=\sum_n(\mathbf{q}_1\times\mathbf{e}_n)\langle(\mathscr{D}^*\mathbf{Z}_n\cdot\mathbf{G}_{mk}\rangle\\&=\sum_n(\mathbf{q}_1\times\mathbf{e}_n)\langle\mathbf{Z}_n\cdot\mathscr{D}\mathbf{G}_{mk}\rangle\\&=-\sum_n(\mathbf{q}_1\times\mathbf{e}_n)\langle\mathbf{Z}_n\cdot(2\eta_0\,\partial\mathbf{S}_k/\partial x_m+\mathbf{e}_m\times(\mathbf{V}\times\mathbf{S}_k))\rangle\\&=\sum_{n,l,j}\epsilon_{jln}q_{1l}\langle\mathbf{S}_k\cdot(2\eta_0\,\partial\mathbf{Z}_n/\partial x_m-\mathbf{V}\times(\mathbf{e}_m\times\mathbf{Z}_n))\rangle\mathbf{e}_j.\end{aligned}$$

Let us consider the component of (16) parallel to $\mathbf{q}_1$. In terms of the matrix $\mathfrak{D}$ (18.2), the scalar product of (16) and $\mathbf{q}_1$ takes the form

$$2\eta_0|\mathbf{q}_1|^2-2\mathrm{Re}\,\langle\langle\mathfrak{D}\langle\mathbf{b}_0\rangle,\mathbf{b}_0^*\rangle\rangle=0.$$

Hence $\mathrm{Re}\,\lambda_2=0$ by virtue of (18.1) and normalization (7.4). We can now clarify the nature of the condition (16). Differentiating (18.1) in q_{1n} yields

$$\Big(\mathfrak{D}-(\lambda_2+\eta_0|\mathbf{q}_1|^2)\Big)\frac{\partial\langle\mathbf{b}_0\rangle}{\partial q_{1n}}-\left(\frac{\partial\lambda_2}{\partial q_{1n}}+2\eta_0\mathbf{q}_{1n}\right)\langle\mathbf{b}_0\rangle-\sum_{k,m}\langle\mathbf{b}_0\rangle_k(\mathbf{q}_1\delta_m^n+q_{1m}\mathbf{e}_n)\times\langle\mathbf{V}\times\mathbf{G}_{mk}\rangle=0.$$

Scalar multiplying this equation by $\overline{\mathbf{b}_0^*}$, applying (7.4), taking the real part and noting that n is an arbitrary index, we obtain $\nabla_{\mathbf{q}_1}\mathrm{Re}\,\lambda_2=-\mathscr{E}\mathbf{q}_1$. Thus, the relation (16) is equivalent to the condition that 0 is an extreme (on varying $\mathbf{q}_1$) large-scale magnetic field growth rate – that is, an extreme real part of the eigenvalue of the operator of eddy diffusivity $\mathfrak{D}-\eta_0|\mathbf{q}_1|^2$. Of course, we choose the dominant eigenvalue, i.e., the maximum over $\mathbf{q}_1$ growth rate, and then η_0 is the molecular diffusivity for which the large-scale dynamo sets in on decreasing η.

For $j=1$, (5) amounts to the orthogonality condition $\mathbf{q}_1\cdot\langle\mathbf{b}_0\rangle=0$. An immediate consequence of (18.4) is that $\mathfrak{D}^*$ has an eigenvalue zero associated with the eigenvector $\mathbf{q}_1$. It is spurious, being associated with an eigenvector of $\mathfrak{D}$ for which $\mathbf{q}_1\cdot\langle\mathbf{b}_0\rangle\ne0$ (see *Rasskazov et al.* [2018], *Zheligovsky* [2011] for details of solving the eigenvalue problem for $\mathfrak{D}$). Consequently, the operator $\mathfrak{D}^*$ has eigenvalues $-\lambda_2+\eta_0|\mathbf{q}_1|^2$, $\lambda_2+\eta_0|\mathbf{q}_1|^2$ and 0 associated with eigenvectors $\mathbf{h}_1^*=\mathbf{b}_0^*$, $\mathbf{h}_2^*=\overline{\mathbf{b}_0^*}$ and $\mathbf{h}_3^*=\mathbf{q}_1$, respectively; the adjoint operator $\mathfrak{D}$ has eigenvalues $\lambda_2+\eta_0|\mathbf{q}_1|^2$, $-\lambda_2+\eta_0|\mathbf{q}_1|^2$ and 0 associated with eigenvectors $\mathbf{h}_1=\langle\mathbf{b}_0\rangle$, $\mathbf{h}_2=\left\langle\overline{\mathbf{b}}_0\right\rangle$ and a vector that we denote $\mathbf{h}_3$. The bases of the eigenvectors of $\mathfrak{D}$ and $\mathfrak{D}^*$ are biorthogonal, i.e., $\mathbf{h}_k\cdot\overline{\mathbf{h}_j^*}=0$ for $k\ne j$, and this scalar product is non-zero for $k=j$ (whereby the normalization (7.4) is always possible).

Equations (18.1), (18.3) and (16) are homogeneous in $\mathbf{q}_1$: upon dividing (18.1) and (18.3) by $|\mathbf{q}_1|^2$, and (16) by $|\mathbf{q}_1|$, none of them involves the length of $\mathbf{q}_1$ (except for in the product $|\mathbf{q}_1|^{-2}\lambda_2$). Therefore, at least in principle, this system can be solved as follows: (i) we find eigenvectors $\langle\mathbf{b}_0\rangle$ of $\mathfrak{D}$ and $\mathbf{b}_0^*$ of the transposed matrix $\mathfrak{D}^*$ associated with non-zero complex conjugate eigenvalues as functions of η_0 and $\mathbf{q}_1/|\mathbf{q}_1|$; (ii) we find $\mathbf{q}_1/|\mathbf{q}_1|$ as a function of η_0 from (16); (iii) we equate η_0 to the real part of the eigenvalue of the matrix $|\mathbf{q}_1|^{-2}\mathfrak{D}$ obtained in step (ii), and solve the resultant equation for η_0; (iv) finally, $|\mathbf{q}_1|^{-2}\lambda_2$ is equal to the imaginary part of the eigenvalue of the matrix $|\mathbf{q}_1|^{-2}\mathfrak{D}$. At this stage we thus fully determine η_0, the eigenvectors $\langle\mathbf{b}_0\rangle$ and $\mathbf{b}_0^*$ (up to arbitrary individual normalizations, that are irrelevant provided the condition (7.4), equivalent to $\langle\mathbf{b}_0\rangle\cdot\overline{\mathbf{b}_0^*}=1$, is imposed), the direction $\mathbf{q}_1/|\mathbf{q}_1|$, and the imaginary value $|\mathbf{q}_1|^{-2}\lambda_2$. We lack an equation for determining the length $|\mathbf{q}_1|$: this scaling is controlled by the term $-\vartheta^2\nabla^2\mathbf{b}$ in the original eigenvalue problem (1), but neither the solvability conditions at this order, nor (16) involve contributions from this term.

By (17.1) and (17.2),

$$\mathbf{b}_2=\sum_k\Big(\langle\mathbf{b}_2\rangle_k\mathbf{S}_k+\langle\mathbf{b}_0\rangle_k\Big((\lambda_2+\eta_0|\mathbf{q}_1|^2)\mathbf{S}_k^{(1)}+\mathbf{S}_k^{(2)}+\sum_{m,n}q_{1m}q_{1n}\mathbf{S}_{nmk}^{(3)}\Big)\Big)$$
$$+\,\mathrm{i}\sum_{k,m}(q_{2m}\langle\mathbf{b}_0\rangle_k+q_{1m}\langle\mathbf{b}_1\rangle_k)\mathbf{G}_{mk}, \tag{19.1}$$

$$\mathbf{b}_2^*=\langle\mathbf{b}_2^*\rangle+\mathrm{i}\sum_{m,k,n}\epsilon_{mkn}(q_{1m}\langle\mathbf{b}_1^*\rangle_k+q_{2m}b_{0k}^*)\mathbf{Z}_n+\sum_{m,k,n,j}\epsilon_{mkn}q_{1m}q_{1j}b_{0k}^*\mathbf{Z}_{jn}^{(1)}, \tag{19.2}$$

where parity-antiinvariant zero-mean functions $\mathbf{S}_k^{(1)}(\mathbf{x})$, $\mathbf{S}_k^{(2)}(\mathbf{x})$, $\mathbf{S}_{nmk}^{(3)}(\mathbf{x})$ and $\mathbf{Z}_{jn}^{(1)}(\mathbf{x})$ solve the auxiliary problems

$$\mathscr{D}\mathbf{S}_k^{(1)}=\mathbf{S}_k-\mathbf{e}_k,$$
$$\mathscr{D}\mathbf{S}_k^{(2)}=\nabla^2\mathbf{S}_k,$$
$$\mathscr{D}\mathbf{S}_{nmk}^{(3)}=2\eta_0\,\partial\mathbf{G}_{mk}/\partial x_n+\mathbf{e}_n\times(\mathbf{V}\times\mathbf{G}_{mk}-\langle\mathbf{V}\times\mathbf{G}_{mk}\rangle),$$
$$\mathscr{D}^*\mathbf{Z}_{jn}^{(1)}=2\eta_0\,\partial\mathbf{Z}_n/\partial x_j-\mathbf{V}\times(\mathbf{e}_j\times\mathbf{Z}_n)+\sum_{l=1}^{3}\Big\langle\mathbf{V}\times\mathbf{G}_{jl}\Big\rangle_n\mathbf{e}_l.$$

Upon substituting (13), (14) and (19), and applying (15), the extremality condition (17.3) takes the form

$$\mathscr{E}\mathbf{q}_2=\sum_{k,m}\Big(\Big(\mathbf{q}_1\times\mathrm{Re}\,(\langle\mathbf{b}_0\rangle_k\Big\langle\overline{\mathbf{b}_1^*}\Big\rangle+\langle\mathbf{b}_1\rangle_k\overline{\mathbf{b}_0^*})\Big)\cdot\langle\mathbf{V}\times\mathbf{G}_{mk}\rangle\,\mathbf{e}_m$$
$$+\,q_{1m}\,\mathrm{Re}\,(\langle\mathbf{b}_0\rangle_k\Big\langle\overline{\mathbf{b}_1^*}\Big\rangle+\langle\mathbf{b}_1\rangle_k\overline{\mathbf{b}_0^*})\times\langle\mathbf{V}\times\mathbf{G}_{mk}\rangle\Big). \tag{20}$$

6. Order ϑ^3 equations

At order ϑ^3, power series expansions of (1), (2), (6) and (4) yield, respectively, the equations

$$\mathscr{D}\mathbf{b}_3 + 2\mathrm{i}\eta_0((\mathbf{q}_1\cdot\nabla)\mathbf{b}_2 + (\mathbf{q}_2\cdot\nabla)\mathbf{b}_1 + (\mathbf{q}_3\cdot\nabla)\mathbf{b}_0) - \nabla^2\mathbf{b}_1 - 2\mathrm{i}(\mathbf{q}_1\cdot\nabla)\mathbf{b}_0 + \mathrm{i}\mathbf{q}_1\times(\mathbf{V}\times\mathbf{b}_2)$$
$$+\mathrm{i}\mathbf{q}_2\times(\mathbf{V}\times\mathbf{b}_1) + \mathrm{i}\mathbf{q}_3\times(\mathbf{V}\times\mathbf{b}_0) = (\lambda_3 + 2\eta_0\,\mathbf{q}_1\cdot\mathbf{q}_2)\mathbf{b}_0 + (\lambda_2 + \eta_0|\mathbf{q}_1|^2)\mathbf{b}_1, \tag{21.1}$$
$$\mathscr{D}^*\mathbf{b}^*_3 + 2\mathrm{i}\eta_0((\mathbf{q}_1\cdot\nabla)\mathbf{b}^*_2 + (\mathbf{q}_2\cdot\nabla)\mathbf{b}^*_1) - \nabla^2\mathbf{b}^*_1 - \mathrm{i}\mathbf{V}\times(\mathbf{q}_1\times\mathbf{b}^*_2 + \mathbf{q}_2\times\mathbf{b}^*_1 + \mathbf{q}_3\times\mathbf{b}^*_0)$$
$$= (\overline{\lambda}_3 + 2\eta_0\,\mathbf{q}_1\cdot\mathbf{q}_2)\mathbf{b}^*_0 + (\overline{\lambda}_2 + \eta_0|\mathbf{q}_1|^2)\mathbf{b}^*_1, \tag{21.2}$$
$$2\eta_0\left(\mathbf{q}_3 + \sum_m \mathrm{Im}\left\langle \frac{\partial\mathbf{b}_0}{\partial x_m}\cdot\overline{\mathbf{b}^*_3} + \frac{\partial\mathbf{b}_1}{\partial x_m}\cdot\overline{\mathbf{b}^*_2} + \frac{\partial\mathbf{b}_2}{\partial x_m}\cdot\overline{\mathbf{b}^*_1}\right\rangle \mathbf{e}_m\right) - 2\left(\mathbf{q}_1 + \sum_m \mathrm{Im}\left\langle \frac{\partial\mathbf{b}_0}{\partial x_m}\cdot\overline{\mathbf{b}^*_1}\right\rangle \mathbf{e}_m\right)$$
$$+\,\mathrm{Im}\left\langle(\mathbf{V}\times\mathbf{b}_0)\times\overline{\mathbf{b}^*_3} + (\mathbf{V}\times\mathbf{b}_1)\times\overline{\mathbf{b}^*_2} + (\mathbf{V}\times\mathbf{b}_2)\times\overline{\mathbf{b}^*_1} + (\mathbf{V}\times\mathbf{b}_3)\times\overline{\mathbf{b}^*_0}\right\rangle = 0, \tag{21.3}$$
$$\langle\langle\mathbf{b}_0,\mathbf{b}^*_3\rangle\rangle + \langle\langle\mathbf{b}_2,\mathbf{b}^*_1\rangle\rangle + \langle\langle\mathbf{b}_1,\mathbf{b}^*_2\rangle\rangle + \langle\langle\mathbf{b}_3,\mathbf{b}^*_0\rangle\rangle = 0. \tag{21.4}$$

Again we use first the solvability conditions for (21.1) and (21.2). Averaging reduces (21.1) to

$$\mathrm{i}\langle\mathbf{q}_1\times(\mathbf{V}\times\mathbf{b}_2) + \mathbf{q}_2\times(\mathbf{V}\times\mathbf{b}_1)\rangle = (\lambda_3 + 2\eta_0\,\mathbf{q}_1\cdot\mathbf{q}_2)\langle\mathbf{b}_0\rangle + (\lambda_2 + \eta_0|\mathbf{q}_1|^2)\langle\mathbf{b}_1\rangle.$$

Upon substituting (13), (14) and (19), the solvability condition for (21.1) becomes

$$\mathfrak{D}\langle\mathbf{b}_1\rangle - \sum_{m,k}\langle\mathbf{b}_0\rangle_k(q_{1m}\mathbf{q}_2 + q_{2m}\mathbf{q}_1)\times\langle\mathbf{V}\times\mathbf{G}_{mk}\rangle = (\lambda_3 + 2\eta_0\,\mathbf{q}_1\cdot\mathbf{q}_2)\langle\mathbf{b}_0\rangle + (\lambda_2 + \eta_0|\mathbf{q}_1|^2)\langle\mathbf{b}_1\rangle. \tag{22}$$

Scalar multiplying it by $\overline{\mathbf{b}^*_0}$, we find

$$\lambda_3 = -2\eta_0\,\mathbf{q}_1\cdot\mathbf{q}_2 - \sum_{m,k}\langle\mathbf{b}_0\rangle_k\Big((q_{1m}\mathbf{q}_2 + q_{2m}\mathbf{q}_1)\times\langle\mathbf{V}\times\mathbf{G}_{mk}\rangle\Big)\cdot\overline{\mathbf{b}^*_0}. \tag{23}$$

Here the real part of the r.h.s. is equal to $-\mathbf{q}_2\cdot\mathscr{E}\mathbf{q}_1$, and thus $\mathrm{Re}\,\lambda_3 = 0$ by (16). Scalar multiplying (22) by $\mathbf{q}_1$ yields a relation equivalent to (5) for $j = 2$. To solve (22), we exploit the biorthogonality of the bases of the eigenvectors of $\mathfrak{D}$ and $\mathfrak{D}^*$, and find

$$\langle\mathbf{b}_1\rangle = \mu_1\langle\mathbf{b}_0\rangle - \sum_{m,k}\langle\mathbf{b}_0\rangle_k\frac{\Big((q_{1m}\mathbf{q}_2 + q_{2m}\mathbf{q}_1)\times\langle\mathbf{V}\times\mathbf{G}_{mk}\rangle\Big)\cdot\mathbf{b}^*_0}{2\lambda_2}\left\langle\overline{\mathbf{b}}_0\right\rangle - \frac{\langle\mathbf{b}_0\rangle\cdot\mathbf{q}_2}{\mathbf{h}_3\cdot\mathbf{q}_1}\mathbf{h}_3, \tag{24}$$

where μ_1 is a constant. The solvability condition for (21.2) is

$$\sum_j\Big\langle\Big(2\mathrm{i}\eta_0((\mathbf{q}_2\cdot\nabla)\mathbf{b}^*_1 + (\mathbf{q}_1\cdot\nabla)\mathbf{b}^*_2) - \mathrm{i}\mathbf{V}\times(\mathbf{q}_1\times\mathbf{b}^*_2 + \mathbf{q}_2\times\mathbf{b}^*_1)\Big)\cdot\mathbf{S}_j\Big\rangle\,\mathbf{e}_j$$
$$= (\overline{\lambda}_3 + 2\eta_0\,\mathbf{q}_1\cdot\mathbf{q}_2)\mathbf{b}^*_0 + (\overline{\lambda}_2 + \eta_0|\mathbf{q}_1|^2)\langle\mathbf{b}^*_1\rangle,$$

or, upon substituting (13), (14) and (19), and using the identity (15)),

$$\mathfrak{D}^*\langle\mathbf{b}^*_1\rangle + \sum_{l,j}\Big(\Big((q_{1l}\mathbf{q}_2 + q_{2l}\mathbf{q}_1)\times\mathbf{b}^*_0\Big)\cdot\Big\langle\mathbf{V}\times\mathbf{G}_{lj}\Big\rangle\Big)\mathbf{e}_j = (\overline{\lambda}_3 + 2\eta_0\,\mathbf{q}_1\cdot\mathbf{q}_2)\mathbf{b}^*_0 + (\overline{\lambda}_2 + \eta_0|\mathbf{q}_1|^2)\langle\mathbf{b}^*_1\rangle. \tag{25}$$

The scalar product of this equation with $\left\langle\overline{\mathbf{b}}_0\right\rangle$ is equivalent to (23). The solution to (25) is

$$\langle\mathbf{b}^*_1\rangle = \mu^*_1\mathbf{b}^*_0 + \sum_{l,j}\Big(\Big((q_{1l}\mathbf{q}_2 + q_{2l}\mathbf{q}_1)\times\mathbf{b}^*_0\Big)\cdot\Big\langle\mathbf{V}\times\mathbf{G}_{lj}\Big\rangle\Big)\left(-\frac{\langle\mathbf{b}_0\rangle_j\overline{\mathbf{b}^*_0}}{2\lambda_2} + \frac{h_{3,j}\mathbf{q}_1}{(\overline{\lambda}_2 + \eta_0|\mathbf{q}_1|^2)\mathbf{h}_3\cdot\mathbf{q}_1}\right), \tag{26}$$

where μ_1^* is a constant. The eigenfunctions can be multiplied by any linear function of ϑ; this changes arbitrarily the coefficients μ_1 and μ_1^*. In view of (9), (13) and (14), the normalization condition (12.4) reduces to $\langle\mathbf{b}_0\rangle\cdot\left\langle\overline{\mathbf{b}_1^*}\right\rangle+\langle\mathbf{b}_1\rangle\cdot\overline{\mathbf{b}_0^*}=0$. Substituting here the expressions (24) and (26), and employing the biorthogonality of the bases $\{\mathbf{h}_k\}$ and $\{\mathbf{h}_j^*\}$ transforms this equation into $\mu_1+\overline{\mu_1^*}=0$. Without any loss of generality we henceforth assume $\mu_1=\mu_1^*=0$.

The solvability condition for the problem (20) in $\mathbf{q}_2$ amounts to the orthogonality of the inhomogeneous term in (20) to $\mathbf{q}_1$ (as discussed at the end of section 4), which is equivalent to the equality

$$-2\mathrm{Re}\Big(\mathcal{D}\langle\mathbf{b}_0\rangle\cdot\left\langle\overline{\mathbf{b}_1^*}\right\rangle+\mathcal{D}\langle\mathbf{b}_1\rangle\cdot\overline{\mathbf{b}_0^*}\Big)=0.$$

Substituting here $\mathcal{D}\langle\mathbf{b}_0\rangle$ (18.1) and $\mathcal{D}\langle\mathbf{b}_1\rangle$ (22), using the normalization conditions (7.4) and (12.4), and relying on (23), we find that this equality holds true.

Upon dividing (22) and (25) by $|\mathbf{q}_1|^2$, and (20) by $|\mathbf{q}_1|$, we obtain equations that involve the direction $\mathbf{q}_1/|\mathbf{q}_1|$ and the vector $\mathbf{q}_2/|\mathbf{q}_1|$, but not the length $|\mathbf{q}_1|$. Hence, we solve this system of equations as follows: (*i*) We have expressed $\langle\mathbf{b}_1\rangle$ and $\left\langle\mathbf{b}_1^*\right\rangle$ as functions of $\mathbf{q}_2/|\mathbf{q}_1|$ and $\lambda_2/|\mathbf{q}_1|^2$. (*ii*) We find $\mathbf{q}_2/|\mathbf{q}_1|$ from the extremality condition (20) up to an unknown additive term $\beta_2\mathbf{q}_1/|\mathbf{q}_1|$ (cf. (16)). (*iii*) The imaginary value $\lambda_3/|\mathbf{q}_1|^2$ is determined by (23). We still lack an equation for determining the length $|\mathbf{q}_1|$.

Next we find solutions to equations the (21.1) and (21.2):

$$\begin{aligned}\mathbf{b}_3=&\sum_k\Big(\langle\mathbf{b}_3\rangle_k\mathbf{S}_k+\Big(\langle\mathbf{b}_0\rangle_k(\lambda_3+2\eta_0\mathbf{q}_1\cdot\mathbf{q}_2)+\langle\mathbf{b}_1\rangle_k(\lambda_2+\eta_0|\mathbf{q}_1|^2)\Big)\mathbf{S}_k^{(1)}+\langle\mathbf{b}_1\rangle_k\mathbf{S}_k^{(2)}\Big)\\&+\sum_{k,m,n}\Big(\langle\mathbf{b}_0\rangle_k(q_{1n}q_{2m}+q_{2n}q_{1m})+\langle\mathbf{b}_1\rangle_kq_{1n}q_{1m}\Big)\mathbf{S}_{nmk}^{(3)}\\&+\mathrm{i}\sum_{k,m}\Big((q_{3m}\langle\mathbf{b}_0\rangle_k+q_{2m}\langle\mathbf{b}_1\rangle_k+q_{1m}\langle\mathbf{b}_2\rangle_k)\mathbf{G}_{mk}\\&\qquad+q_{1m}\langle\mathbf{b}_0\rangle_k\Big((\lambda_2+\eta_0|\mathbf{q}_1|^2)\mathbf{G}_{mk}^{(1)}+\mathbf{G}_{mk}^{(2)}+\sum_{j,n}q_{1j}q_{1n}\mathbf{G}_{jnmk}^{(3)}\Big)\Big),\end{aligned}\tag{27.1}$$

$$\begin{aligned}\mathbf{b}_3^*=&\langle\mathbf{b}_3^*\rangle+\mathrm{i}\sum_{m,k,n}\epsilon_{mkn}(q_{1m}\langle\mathbf{b}_2^*\rangle_k+q_{2m}\langle\mathbf{b}_1^*\rangle_k+q_{3m}b_{0k}^*)\mathbf{Z}_n\\&+\sum_{m,k,n,j}\epsilon_{mkn}\Big(\Big(b_{0k}^*(q_{1m}q_{2j}+q_{2m}q_{1j})+\langle\mathbf{b}_1^*\rangle_kq_{1m}q_{1j}\Big)\mathbf{Z}_{jn}^{(1)}+\mathrm{i}\sum_p b_{0k}^*q_{1m}q_{1j}q_{1p}\mathbf{Z}_{jnp}^{(2)}\Big)\\&+\mathrm{i}\sum_{m,k,n}\epsilon_{mkn}q_{1m}b_{0k}^*\Big(\mathbf{Z}_n^{(3)}+(\overline{\lambda}_2+\eta_0|\mathbf{q}_1|^2)\mathbf{Z}_n^{(4)}\Big),\end{aligned}\tag{27.2}$$

where we refer to parity-invariant solutions to the following auxiliary problems:

$$\begin{aligned}&\mathscr{D}\mathbf{G}_{mk}^{(1)}+2\eta_0\,\partial\mathbf{S}_k^{(1)}/\partial x_m+\mathbf{e}_m\times(\mathbf{V}\times\mathbf{S}_k^{(1)})-\mathbf{G}_{mk}=0;\\&\mathscr{D}\mathbf{G}_{mk}^{(2)}+2\eta_0\,\partial\mathbf{S}_k^{(2)}/\partial x_m+\mathbf{e}_m\times(\mathbf{V}\times\mathbf{S}_k^{(2)})-2\partial\mathbf{S}_k/\partial x_m-\nabla^2\mathbf{G}_{mk}=0;\\&\mathscr{D}\mathbf{G}_{jnmk}^{(3)}+2\eta_0\,\partial\mathbf{S}_{nmk}^{(3)}/\partial x_j+\mathbf{e}_j\times(\mathbf{V}\times\mathbf{S}_{nmk}^{(3)})=0;\\&\mathscr{D}^*\mathbf{Z}_{jnp}^{(2)}=2\eta_0\,\partial\mathbf{Z}_{jn}^{(1)}/\partial x_p-\mathbf{V}\times(\mathbf{e}_p\times\mathbf{Z}_{jn}^{(1)});\\&\mathscr{D}^*\mathbf{Z}_n^{(3)}=\nabla^2\mathbf{Z}_n;\\&\mathscr{D}^*\mathbf{Z}_n^{(4)}=\mathbf{Z}_n.\end{aligned}$$

Upon substituting (13), (19) and (27), and applying (15), the extremality condition (21.3) reduces to

$$\mathscr{E}\mathbf{q}_3-\sum_{l,j}\left\langle\mathbf{V}\times\mathbf{G}_{lj}\right\rangle\cdot\mathrm{Re}\Big(\langle\mathbf{b}_0\rangle_j\overline{(\mathbf{q}_1\times\langle\mathbf{b}^*_2\rangle+\mathbf{q}_2\times\langle\mathbf{b}^*_1\rangle)}+\langle\mathbf{b}_1\rangle_j\overline{(\mathbf{q}_1\times\langle\mathbf{b}^*_1\rangle+\mathbf{q}_2\times\mathbf{b}^*_0)}+\langle\mathbf{b}_2\rangle_j\mathbf{q}_1\times\overline{\mathbf{b}^*_0}\Big)\,\mathbf{e}_l$$
$$+\sum_{k,m}\langle\mathbf{V}\times\mathbf{G}_{mk}\rangle\times\mathrm{Re}\Big((q_{2m}\langle\mathbf{b}_1\rangle_k+q_{1m}\langle\mathbf{b}_2\rangle_k)\overline{\mathbf{b}^*_0}+(q_{2m}\langle\mathbf{b}_0\rangle_k+q_{1m}\langle\mathbf{b}_1\rangle_k)\overline{\langle\mathbf{b}^*_1\rangle}+q_{1m}\langle\mathbf{b}_0\rangle_k\overline{\langle\mathbf{b}^*_2\rangle}\Big)$$
$$+2\eta_0\sum_l\mathrm{Re}\Big\langle-\sum_{j,m,k,n}\epsilon_{mkn}q_{1m}\overline{b^*_{0k}}\langle\mathbf{b}_0\rangle_j\frac{\partial\mathbf{S}_j}{\partial x_l}\cdot\Big(\mathbf{Z}^{(3)}_n+(\lambda_2+\eta_0|\mathbf{q}_1|^2)\mathbf{Z}^{(4)}_n+\sum_{p,r}q_{1r}q_{1p}\,\mathbf{Z}^{(2)}_{rnp}\Big)$$
$$+\Big(\sum_{k,m}q_{1m}\langle\mathbf{b}_0\rangle_k\frac{\partial\mathbf{G}_{mk}}{\partial x_l}\Big)\cdot\sum_{m,k,n,r}\epsilon_{mkn}\,q_{1m}q_{1r}\,\overline{b^*_{0k}}\,\mathbf{Z}^{(1)}_{rn}$$
$$-\Big(\frac{\partial}{\partial x_l}\sum_j\langle\mathbf{b}_0\rangle_j\Big((\lambda_2+\eta_0|\mathbf{q}_1|^2)\mathbf{S}^{(1)}_j+\mathbf{S}^{(2)}_j+\sum_{m,n}q_{1m}q_{1n}\mathbf{S}^{(3)}_{nmj}\Big)\Big)\cdot\sum_{m,k,n}\epsilon_{mkn}\,q_{1m}\,\overline{b^*_{0k}}\,\mathbf{Z}_n\Big\rangle\,\mathbf{e}_l$$
$$-2\Big(\mathbf{q}_1-\sum_{l,j,m,k,n}\epsilon_{mkn}\,q_{1m}\,\mathrm{Re}\Big(\overline{b^*_{0k}}\langle\mathbf{b}_0\rangle_j\Big)\Big\langle\frac{\partial\mathbf{S}_j}{\partial x_l}\cdot\mathbf{Z}_n\Big\rangle\,\mathbf{e}_l\Big)$$
$$+\mathrm{Re}\,\Big\langle-\Big(\mathbf{V}\times\sum_j\langle\mathbf{b}_0\rangle_j\mathbf{S}_j\Big)\times\sum_{m,k,n}\epsilon_{mkn}q_{1m}\,\overline{b^*_{0k}}\Big(\mathbf{Z}^{(3)}_n+(\lambda_2+\eta_0|\mathbf{q}_1|^2)\mathbf{Z}^{(4)}_n+\sum_{p,r}q_{1r}q_{1p}\,\mathbf{Z}^{(2)}_{rnp}\Big)$$
$$+\Big(\mathbf{V}\times\sum_{k,m}q_{1m}\langle\mathbf{b}_0\rangle_k\mathbf{G}_{mk}\Big)\times\sum_{m,k,n,p}\epsilon_{mkn}\,q_{1m}q_{1p}\,\overline{b^*_{0k}}\,\mathbf{Z}^{(1)}_{pn}$$
$$-\Big(\mathbf{V}\times\sum_j\langle\mathbf{b}_0\rangle_j\Big((\lambda_2+\eta_0|\mathbf{q}_1|^2)\mathbf{S}^{(1)}_j+\mathbf{S}^{(2)}_j+\sum_{m,n}q_{1m}q_{1n}\mathbf{S}^{(3)}_{nmj}\Big)\Big)\times\sum_{m,k,n}\epsilon_{mkn}\,q_{1m}\,\overline{b^*_{0k}}\,\mathbf{Z}_n$$
$$+\Big(\mathbf{V}\times\sum_{k,m}q_{1m}\langle\mathbf{b}_0\rangle_k\Big((\lambda_2+\eta_0|\mathbf{q}_1|^2)\mathbf{G}^{(1)}_{mk}+\mathbf{G}^{(2)}_{mk}+\sum_{j,n}q_{1j}q_{1n}\mathbf{G}^{(3)}_{jnmk}\Big)\Big)\times\overline{\mathbf{b}^*_0}\Big\rangle=0. \qquad (28)$$

Scalar multiplying (28) by $\mathbf{q}_1$ yields the solvability condition for (28):

$$\sum_{l,j}\left\langle\mathbf{V}\times\mathbf{G}_{lj}\right\rangle\cdot\Big((q_{1l}\mathbf{q}_2+q_{2l}\mathbf{q}_1)\times\mathrm{Re}(\langle\mathbf{b}_1\rangle_j\overline{\mathbf{b}^*_0}-\langle\mathbf{b}_0\rangle_j\overline{\langle\mathbf{b}^*_1\rangle})\Big)$$
$$+2\mathrm{Re}\Big((\lambda_2+\eta_0|\mathbf{q}_1|^2)\sum_{j,m,k,n,p}\epsilon_{mkn}q_{1m}q_{1j}\overline{b^*_{0k}}\langle\mathbf{b}_0\rangle_p\Big\langle\mathbf{S}_p\cdot\mathbf{Z}^{(1)}_{jn}+\mathbf{G}^{(1)}_{jp}\cdot\mathbf{Z}_n\Big\rangle-(\lambda_3+2\eta_0\,\mathbf{q}_1\cdot\mathbf{q}_2)\langle\mathbf{b}_1\rangle\cdot\overline{\mathbf{b}^*_0}\Big)$$
$$+2\eta_0\sum_l\mathrm{Re}\Big\langle-\sum_{j,m,k,n}\epsilon_{mkn}q_{1m}\overline{b^*_{0k}}\langle\mathbf{b}_0\rangle_j\frac{\partial\mathbf{S}_j}{\partial x_l}\cdot\Big(\mathbf{Z}^{(3)}_n+(\lambda_2+\eta_0|\mathbf{q}_1|^2)\mathbf{Z}^{(4)}_n+\sum_{p,r}q_{1r}q_{1p}\,\mathbf{Z}^{(2)}_{rnp}\Big)$$
$$+\Big(\sum_{k,m}q_{1m}\langle\mathbf{b}_0\rangle_k\frac{\partial\mathbf{G}_{mk}}{\partial x_l}\Big)\cdot\sum_{m,k,n,r}\epsilon_{mkn}\,q_{1m}q_{1r}\,\overline{b^*_{0k}}\,\mathbf{Z}^{(1)}_{rn}$$
$$-\Big(\frac{\partial}{\partial x_l}\sum_j\langle\mathbf{b}_0\rangle_j\Big((\lambda_2+\eta_0|\mathbf{q}_1|^2)\mathbf{S}^{(1)}_j+\mathbf{S}^{(2)}_j+\sum_{m,n}q_{1m}q_{1n}\mathbf{S}^{(3)}_{nmj}\Big)\Big)\cdot\sum_{m,k,n}\epsilon_{mkn}\,q_{1m}\,\overline{b^*_{0k}}\,\mathbf{Z}_n\Big\rangle q_{1l}$$
$$-2\Big(|\mathbf{q}_1|^2-\sum_{l,j,m,k,n}\epsilon_{mkn}\,q_{1m}\,\mathrm{Re}\Big(\overline{b^*_{0k}}\langle\mathbf{b}_0\rangle_j\Big)\Big\langle\frac{\partial\mathbf{S}_j}{\partial x_l}\cdot\mathbf{Z}_n\Big\rangle q_{1l}\Big)$$
$$+\mathrm{Re}\Big\langle-\Big(\mathbf{V}\times\sum_j\langle\mathbf{b}_0\rangle_j\mathbf{S}_j\Big)\times\sum_{m,k,n}\epsilon_{mkn}q_{1m}\,\overline{b^*_{0k}}\Big(\mathbf{Z}^{(3)}_n+(\lambda_2+\eta_0|\mathbf{q}_1|^2)\mathbf{Z}^{(4)}_n+\sum_{p,r}q_{1r}q_{1p}\,\mathbf{Z}^{(2)}_{rnp}\Big)$$
$$+\Big(\mathbf{V}\times\sum_{k,m}q_{1m}\langle\mathbf{b}_0\rangle_k\mathbf{G}_{mk}\Big)\times\sum_{m,k,n,p}\epsilon_{mkn}\,q_{1m}q_{1p}\,\overline{b^*_{0k}}\,\mathbf{Z}^{(1)}_{pn}$$

$$-\Big(\mathbf{V}\times\sum_j\langle\mathbf{b}_0\rangle_j\Big((\lambda_2+\eta_0|\mathbf{q}_1|^2)\mathbf{S}^{(1)}_j+\mathbf{S}^{(2)}_j+\sum_{m,n}q_{1m}q_{1n}\mathbf{S}^{(3)}_{nmj}\Big)\Big)\times\sum_{m,k,n}\epsilon_{mkn}\,q_{1m}\,\overline{b^*_{0k}}\,\mathbf{Z}_n$$
$$+\Big(\mathbf{V}\times\sum_{k,m}q_{1m}\langle\mathbf{b}_0\rangle_k\Big((\lambda_2+\eta_0|\mathbf{q}_1|^2)\mathbf{G}^{(1)}_{mk}+\mathbf{G}^{(2)}_{mk}+\sum_{j,n}q_{1j}q_{1n}\mathbf{G}^{(3)}_{jnmk}\Big)\Big)\times\overline{\mathbf{b}^*_0}\Big\rangle\cdot\mathbf{q}_1=0. \quad (29)$$

At this stage $\mathbf{q}_2$ is known up to the factor $|\mathbf{q}_1|^{-1}$ and an additive term proportional to $\mathbf{q}_1$. Expressions (24) and (26) reveal that $\langle\mathbf{b}_1\rangle$ and $\left\langle\mathbf{b}^*_1\right\rangle$ depend on $\mathbf{q}_2$, but it is easy to check (relying on (18.1), (18.2) and the biorthogonality of the bases of the eigenvectors of the operators $\mathfrak{D}$ and $\mathfrak{D}^*$) that they are invariant under the transformation $\mathbf{q}_2\to\mathbf{q}_2+\beta\mathbf{q}_1$ for any β. The remaining potentially affected terms in (29) are the first term involving $\mathbf{q}_2$ explicitly, and, by virtue of (23), $\lambda_3+2\eta_0\,\mathbf{q}_1\cdot\mathbf{q}_2$. For a real β, the total change in the l.h.s. of (29) due to the transformation $\mathbf{q}_2\to\mathbf{q}_2+\beta\mathbf{q}_1$ is

$$2\beta\mathrm{Re}\Big(\mathfrak{D}\langle\mathbf{b}_1\rangle\cdot\overline{\mathbf{b}^*_0}-\mathfrak{D}\langle\mathbf{b}_0\rangle\cdot\left\langle\overline{\mathbf{b}^*_1}\right\rangle-2(\mathfrak{D}\langle\mathbf{b}_0\rangle\cdot\overline{\mathbf{b}^*_0})\langle\mathbf{b}_1\rangle\cdot\overline{\mathbf{b}^*_0}\Big)$$
$$=-2\beta\mathrm{Re}\Big((\lambda_2+\eta_0|\mathbf{q}_1|^2)(\langle\mathbf{b}_1\rangle\cdot\overline{\mathbf{b}^*_0}+\langle\mathbf{b}_0\rangle\cdot\left\langle\overline{\mathbf{b}^*_1}\right\rangle)\Big)=0$$

(we have employed relations (18.1)–(18.3), (23) and the normalization conditions at orders ϑ^0 and ϑ^1). Thus, the solvability condition (29) is not altered by the unknown additive component of $\mathbf{q}_2$, parallel to $\mathbf{q}_1$. All the quantities involved in (29) are known completely or up to factors that are powers of $|\mathbf{q}_1|$. Substituting the values of these quantities transforms (29) into an inhomogeneous equation in $|\mathbf{q}_1|$, involving terms proportional to $|\mathbf{q}_1|^2$ and $|\mathbf{q}_1|^4$. We determine from it the length $|\mathbf{q}_1|$. Now (28) yields $\mathbf{q}_3$ up to an additive term that is a multiple of $\mathbf{q}_1$.

7. Order ϑ^4 equations

At order ϑ^4, (1), (2), (6) and (4) yield, respectively, the equations

$$\mathscr{D}\mathbf{b}_4+2\mathrm{i}\eta_0((\mathbf{q}_1\cdot\nabla)\mathbf{b}_3+(\mathbf{q}_2\cdot\nabla)\mathbf{b}_2+(\mathbf{q}_3\cdot\nabla)\mathbf{b}_1+(\mathbf{q}_4\cdot\nabla)\mathbf{b}_0)-\nabla^2\mathbf{b}_2$$
$$-2\mathrm{i}((\mathbf{q}_2\cdot\nabla)\mathbf{b}_0+(\mathbf{q}_1\cdot\nabla)\mathbf{b}_1))+\mathrm{i}\mathbf{q}_1\times(\mathbf{V}\times\mathbf{b}_3)$$
$$+\mathrm{i}\mathbf{q}_2\times(\mathbf{V}\times\mathbf{b}_2)+\mathrm{i}\mathbf{q}_3\times(\mathbf{V}\times\mathbf{b}_1)+\mathrm{i}\mathbf{q}_4\times(\mathbf{V}\times\mathbf{b}_0)$$
$$=(\lambda_4+\eta_0(2\mathbf{q}_1\cdot\mathbf{q}_3+|\mathbf{q}_2|^2)-|\mathbf{q}_1|^2)\mathbf{b}_0+(\lambda_3+2\eta_0\,\mathbf{q}_1\cdot\mathbf{q}_2)\mathbf{b}_1+(\lambda_2+\eta_0|\mathbf{q}_1|^2)\mathbf{b}_2, \quad (30.1)$$
$$\mathscr{D}^*\mathbf{b}^*_4+2\mathrm{i}\eta_0((\mathbf{q}_1\cdot\nabla)\mathbf{b}^*_3+(\mathbf{q}_2\cdot\nabla)\mathbf{b}^*_2+(\mathbf{q}_3\cdot\nabla)\mathbf{b}^*_1)-\nabla^2\mathbf{b}^*_2-2\mathrm{i}(\mathbf{q}_1\cdot\nabla)\mathbf{b}^*_1$$
$$-\mathrm{i}\mathbf{V}\times(\mathbf{q}_1\times\mathbf{b}^*_3+\mathbf{q}_2\times\mathbf{b}^*_2+\mathbf{q}_3\times\mathbf{b}^*_1+\mathbf{q}_4\times\mathbf{b}^*_0)$$
$$=(\overline{\lambda}_4+\eta_0(2\mathbf{q}_1\cdot\mathbf{q}_3+|\mathbf{q}_2|^2)-|\mathbf{q}_1|^2)\mathbf{b}^*_0+(\overline{\lambda}_3+2\eta_0\,\mathbf{q}_1\cdot\mathbf{q}_2)\mathbf{b}^*_1+(\overline{\lambda}_2+\eta_0|\mathbf{q}_1|^2)\mathbf{b}^*_2, \quad (30.2)$$
$$2\eta_0\Big(\mathbf{q}_4+\sum_m\mathrm{Im}\left\langle\frac{\partial\mathbf{b}_0}{\partial x_m}\cdot\overline{\mathbf{b}^*_4}+\frac{\partial\mathbf{b}_1}{\partial x_m}\cdot\overline{\mathbf{b}^*_3}+\frac{\partial\mathbf{b}_2}{\partial x_m}\cdot\overline{\mathbf{b}^*_2}+\frac{\partial\mathbf{b}_3}{\partial x_m}\cdot\overline{\mathbf{b}^*_1}\right\rangle\mathbf{e}_m\Big)$$
$$-2\Big(\mathbf{q}_2+\sum_m\mathrm{Im}\left\langle\frac{\partial\mathbf{b}_0}{\partial x_m}\cdot\overline{\mathbf{b}^*_2}+\frac{\partial\mathbf{b}_1}{\partial x_m}\cdot\overline{\mathbf{b}^*_1}\right\rangle\mathbf{e}_m\Big)+\mathrm{Im}\,\Big\langle(\mathbf{V}\times\mathbf{b}_0)\times\overline{\mathbf{b}^*_4}+(\mathbf{V}\times\mathbf{b}_1)\times\overline{\mathbf{b}^*_3}$$
$$+(\mathbf{V}\times\mathbf{b}_2)\times\overline{\mathbf{b}^*_2}+(\mathbf{V}\times\mathbf{b}_3)\times\overline{\mathbf{b}^*_1}+(\mathbf{V}\times\mathbf{b}_4)\times\overline{\mathbf{b}^*_0}\Big\rangle=0, \quad (30.3)$$
$$\langle\langle\mathbf{b}_0,\mathbf{b}^*_4\rangle\rangle+\langle\langle\mathbf{b}_3,\mathbf{b}^*_1\rangle\rangle+\langle\langle\mathbf{b}_2,\mathbf{b}^*_2\rangle\rangle+\langle\langle\mathbf{b}_1,\mathbf{b}^*_3\rangle\rangle+\langle\langle\mathbf{b}_4,\mathbf{b}^*_0\rangle\rangle=0. \quad (30.4)$$

Averaging (30.1) yields its solvability condition

$$\mathrm{i}\langle\mathbf{q}_1\times(\mathbf{V}\times\mathbf{b}_3)+\mathbf{q}_2\times(\mathbf{V}\times\mathbf{b}_2)+\mathbf{q}_3\times(\mathbf{V}\times\mathbf{b}_1)\rangle$$
$$=(\lambda_4+\eta_0(2\mathbf{q}_1\cdot\mathbf{q}_3+|\mathbf{q}_2|^2)-|\mathbf{q}_1|^2)\langle\mathbf{b}_0\rangle+(\lambda_3+2\eta_0\,\mathbf{q}_1\cdot\mathbf{q}_2)\langle\mathbf{b}_1\rangle+(\lambda_2+\eta_0|\mathbf{q}_1|^2)\langle\mathbf{b}_2\rangle. \quad (31)$$

Upon substitution of the expressions (13), (14.1), (19.1) and (27.1) for $\mathbf{b}_i$, $i = 1,2,3$, this equation takes the form

$$\left(\mathfrak{D} - (\lambda_2 + \eta_0|\mathbf{q}_1|^2)\right)\langle \mathbf{b}_2 \rangle = \mathbf{R}_2 + (\lambda_4 + \eta_0(2\mathbf{q}_1 \cdot \mathbf{q}_3 + |\mathbf{q}_2|^2) - |\mathbf{q}_1|^2)\langle \mathbf{b}_0 \rangle, \tag{32}$$

where

$$\mathbf{R}_2 = \sum_{k,m} \Big(q_{1m} \langle \mathbf{b}_0 \rangle_k \mathbf{q}_1 \times \Big\langle \mathbf{V} \times \Big((\lambda_2 + \eta_0|\mathbf{q}_1|^2)\mathbf{G}^{(1)}_{mk} + \mathbf{G}^{(2)}_{mk} + \sum_{j,n} q_{1j} q_{1n} \mathbf{G}^{(3)}_{jnmk} \Big) \Big\rangle$$
$$+ \Big(\mathbf{q}_1 (q_{3m} \langle \mathbf{b}_0 \rangle_k + q_{2m} \langle \mathbf{b}_1 \rangle_k) + \mathbf{q}_2 (q_{2m} \langle \mathbf{b}_0 \rangle_k + q_{1m} \langle \mathbf{b}_1 \rangle_k) + \mathbf{q}_3 q_{1m} \langle \mathbf{b}_0 \rangle_k \Big) \times \langle \mathbf{V} \times \mathbf{G}_{mk} \rangle \Big)$$
$$+ (\lambda_3 + 2\eta_0\, \mathbf{q}_1 \cdot \mathbf{q}_2) \langle \mathbf{b}_1 \rangle.$$

Scalar multiplying (32) by $\overline{\mathbf{b}_0^*}$, we find

$$\lambda_4 = -\eta_0(2\mathbf{q}_1 \cdot \mathbf{q}_3 + |\mathbf{q}_2|^2) + |\mathbf{q}_1|^2 - \left\langle \overline{\mathbf{b}_0^*} \cdot \mathbf{R}_2 \right\rangle. \tag{33}$$

Since $\mathbf{q}_1$ belongs to the kernel of the operator $\mathscr{E}$ (see (16)), the resultant expression for the growth rate does not involve $\mathbf{q}_3$:

$$\mathrm{Re}\, \lambda_4 = -\overline{\mathbf{b}_0^*} \cdot \Big(\sum_{k,m} \Big(q_{1m} \langle \mathbf{b}_0 \rangle_k \mathbf{q}_1 \times \Big\langle \mathbf{V} \times \Big((\lambda_2 + \eta_0|\mathbf{q}_1|^2)\mathbf{G}^{(1)}_{mk} + \mathbf{G}^{(2)}_{mk} + \sum_{j,n} q_{1j} q_{1n} \mathbf{G}^{(3)}_{jnmk} \Big) \Big\rangle$$
$$+ \Big(\mathbf{q}_1 q_{2m} \langle \mathbf{b}_1 \rangle_k + \mathbf{q}_2 (q_{2m} \langle \mathbf{b}_0 \rangle_k + q_{1m} \langle \mathbf{b}_1 \rangle_k) \Big) \times \langle \mathbf{V} \times \mathbf{G}_{mk} \rangle \Big)$$
$$+ (\lambda_3 + 2\eta_0\, \mathbf{q}_1 \cdot \mathbf{q}_2) \langle \mathbf{b}_1 \rangle \Big) - \eta_0 |\mathbf{q}_2|^2 + |\mathbf{q}_1|^2. \tag{34}$$

Furthermore, it is easy to check that this expression remains invariant under the transformation $\mathbf{q}_2 \to \mathbf{q}_2 + \beta \mathbf{q}_1$ for any real β. Thus, (34) uniquely determines the leading term of the expansion of the growth rate.

Scalar multiplying (31) by $\mathbf{q}_1$ yields (5) for $j = 3$ (given that (5) is satisfied for $j = 2$). We find $\langle \mathbf{b}_2 \rangle$ exploiting the biorthogonality of the bases of the eigenvectors of $\mathfrak{D}$ and $\mathfrak{D}^*$:

$$\langle \mathbf{b}_2 \rangle = \mu_2 \langle \mathbf{b}_0 \rangle - \frac{\mathbf{R}_2 \cdot \mathbf{b}_0^*}{2\lambda_2} \left\langle \overline{\mathbf{b}_0} \right\rangle - \frac{\langle \mathbf{b}_0 \rangle \cdot \mathbf{q}_3 + \langle \mathbf{b}_1 \rangle \cdot \mathbf{q}_2}{\mathbf{h}_3 \cdot \mathbf{q}_1} \mathbf{h}_3,$$

where μ_2 is a constant.

The solvability condition for (30.2) amounts to the orthogonality of the non-homogeneous term in this equation to all $\mathbf{S}_l$:

$$\sum_l \Big\langle \mathbf{S}_l \cdot \Big(2\mathrm{i}\eta_0 ((\mathbf{q}_1 \cdot \nabla)\mathbf{b}_3^* + (\mathbf{q}_2 \cdot \nabla)\mathbf{b}_2^* + (\mathbf{q}_3 \cdot \nabla)\mathbf{b}_1^*) - \nabla^2 \mathbf{b}_2^* - 2\mathrm{i}(\mathbf{q}_1 \cdot \nabla)\mathbf{b}_1^*$$
$$-\mathrm{i}\mathbf{V} \times (\mathbf{q}_1 \times \mathbf{b}_3^* + \mathbf{q}_2 \times \mathbf{b}_2^* + \mathbf{q}_3 \times \mathbf{b}_1^*) - (\overline{\lambda}_2 + \eta_0 |\mathbf{q}_1|^2) \mathbf{b}_2^* \Big) \Big\rangle\, \mathbf{e}_l$$
$$= (\overline{\lambda}_4 + \eta_0(2\mathbf{q}_1 \cdot \mathbf{q}_3 + |\mathbf{q}_2|^2) - |\mathbf{q}_1|^2)\mathbf{b}_0^* + (\overline{\lambda}_3 + 2\eta_0\, \mathbf{q}_1 \cdot \mathbf{q}_2)\langle \mathbf{b}_1^* \rangle.$$

Substituting the expressions (13), (14.2), (19.2) and (27.2) for $\mathbf{b}_i^*$, $i = 1,2,3$, transforms this equation into

$$\left(\mathfrak{D}^* - (\overline{\lambda}_2 + \eta_0|\mathbf{q}_1|^2)\right)\langle \mathbf{b}_2^* \rangle = \mathbf{R}_2^*, \tag{35}$$

where

$$\mathbf{R}_2^* = -\sum_{l,p}\Big(\Big(q_{1p}(\mathbf{q}_2\times\langle\mathbf{b}_1^*\rangle+\mathbf{q}_3\times\mathbf{b}_0^*)+q_{2p}(\mathbf{q}_1\times\langle\mathbf{b}_1^*\rangle+\mathbf{q}_2\times\mathbf{b}_0^*)+q_{3p}\mathbf{q}_1\times\mathbf{b}_0^*\Big)\cdot\Big\langle\mathbf{V}\times\mathbf{G}_{pl}\Big\rangle\Big)\,\mathbf{e}_l$$
$$-\sum_{m,k,n,l}\epsilon_{mkn}\Big\langle\mathbf{S}_l\cdot\Big(-2\eta_0\,b_{0k}^*q_{1m}(\mathbf{q}_1\cdot\nabla)\Big(\sum_{j,p}q_{1j}q_{1p}\,\mathbf{Z}_{jnp}^{(2)}+\mathbf{Z}_n^{(3)}+(\overline{\lambda}_2+\eta_0|\mathbf{q}_1|^2)\mathbf{Z}_n^{(4)}\Big)$$
$$-\sum_j\Big(b_{0k}^*(q_{1m}q_{2j}+q_{2m}q_{1j})+\langle\mathbf{b}_1^*\rangle_k q_{1m}q_{1j}\Big)\nabla^2\mathbf{Z}_{jn}^{(1)}+2b_{0k}^*q_{1m}(\mathbf{q}_1\cdot\nabla)\mathbf{Z}_n$$
$$+\,b_{0k}^*q_{1m}\mathbf{V}\times\Big(\mathbf{q}_1\times\Big(\sum_{j,p}q_{1j}q_{1p}\,\mathbf{Z}_{jnp}^{(2)}+\mathbf{Z}_n^{(3)}+(\overline{\lambda}_2+\eta_0|\mathbf{q}_1|^2)\mathbf{Z}_n^{(4)}\Big)\Big)$$
$$-(\overline{\lambda}_2+\eta_0|\mathbf{q}_1|^2)q_{1m}b_{0k}^*\sum_j q_{1j}\,\mathbf{Z}_{jn}^{(1)}\Big)\Big\rangle\,\mathbf{e}_l+(\overline{\lambda}_3+2\eta_0\,\mathbf{q}_1\cdot\mathbf{q}_2)\langle\mathbf{b}_1^*\rangle+\Big(\overline{\lambda}_4+\eta_0(2\mathbf{q}_1\cdot\mathbf{q}_3+|\mathbf{q}_2|^2)-|\mathbf{q}_1|^2\Big)\,\mathbf{b}$$

The solvability condition for (35), $\mathbf{R}_2^*\cdot\Big\langle\overline{\mathbf{b}_0}\Big\rangle=0$, is equivalent to (33). Due to the biorthogonality of the bases of the eigenvectors of $\mathfrak{D}$ and $\mathfrak{D}^*$, the solution to (35) is

$$\langle\mathbf{b}_2^*\rangle=\mu_2^*\mathbf{b}_0^*+\frac{\mathbf{R}_2^*\cdot\langle\mathbf{b}_0\rangle}{2\lambda_2}\,\overline{\mathbf{b}_0^*}-\frac{\mathbf{R}_2^*\cdot\mathbf{h}_3}{(\overline{\lambda}_2+\eta_0|\mathbf{q}_1|^2)\mathbf{h}_3\cdot\mathbf{q}_1}\mathbf{q}_1,$$

where μ_2^* is a constant. By multiplying the eigenfunctions $\mathbf{b}_2$ and $\mathbf{b}_2^*$ by linear functions of ϑ^2, we can arbitrarily change μ_2 and μ_2^*. The value of the sum $\mu_2+\overline{\mu_2^*}=0$ is prescribed by the normalization condition (17.4), but no other constraints are imposed on the coefficients μ_2 and μ_2^*.

From equations (30.1) and (30.2), we now find the fluctuating parts of $\mathbf{b}_4$ and $\mathbf{b}_4^*$ and substitute the results into (30.3). The solvability condition for the resultant equation yields the unknown so far additive term proportional to $\mathbf{q}_1$ that is involved in $\mathbf{q}_2$. From the equation we obtain $\mathbf{q}_4$ up to an analogous additive term.

Reasoning similarly, we can solve successively the systems of equations at all higher orders and determine all terms in the series (3).

8. Concluding remarks

We have developed a power series asymptotic expansion in $\vartheta=(\eta_0-\eta)^{1/2}$ of magnetic Bloch modes, kinematically generated by a parity-invariant flow and featuring locally maximum growth rates, which stem from the branch of neutral (i.e., associated with the zero eigenvalue of the magnetic induction operator $\mathscr{D}$ (8)) space-periodic modes for $\mathbf{q}=0$ (see Fig. 14 in [*Chertovskih and Zheligovsky*, 2023b]). We have shown that branching occurs at a critical molecular diffusivity η_0 for which the two eigenvalues of the operator of eddy diffusivity are imaginary and complex-conjugate. The associated eigenvalues of the eigenmodes in the offshoot are order ϑ^2 and the first two terms of their expansion (order ϑ^2 and ϑ^3) turn out to be imaginary. The locally maximum (over $\mathbf{q}$) growth rates in the offshoot are order ϑ^4; they are attained for the optimal $\mathbf{q}=\mathrm{O}(\vartheta)$. Thus, the optimal $\mathbf{q}$ and the maximum growth rate are continuous in η, but $\mathbf{q}$ is not differentiable when passing through the critical point η_0 from the branch of the neutral magnetic modes ($\eta>\eta_0$) to the offshoot ($\eta<\eta_0$).

We have expected this instance of branching to be representative. Indeed, numerical estimates reveal that for all the offshoots found *ibid.* the deviations of the optimal wave vectors $\mathbf{q}$ from the constant half-integer $\mathbf{q}$ in the respective main branch are order ϑ, and therefore expansions in power series in ϑ can be constructed for all offshoots. However, Table 1 reveals that the branchings differ in detail. (Recalling that the eigenvalues of the operators of linearization have period one in each component q_m of the Bloch wave vector $\mathbf{q}$, we do not distinguish the component values $\pm 1/2$ in the table and in the text that follows.) For instance, the kinematic dynamo problem for a parity-invariant flow, for which

Table 1. Instances of branching found numerically in [*Chertovskih and Zheligovsky*, 2023b]. F: figures *ibid.* showing the plots; P: the linear stability problem, HD hydrodynamic, MHD magnetohydrodynamic, KD kinematic dynamo; ES: type of the energy spectrum of the state subjected to perturbations; S: the symmetry of the state subjected to perturbations, PI parity-invariant, nPI non-symmetric; ν,η: the molecular viscosity and/or diffusivity, at which branching occurs; $\boldsymbol{q}$: the Bloch wave vector in the main branch; γ: the growth rate of the instability at the branching point, common to both the main branch and offshoot; A: the order of the growth rates in the offshoot, or of the real and imaginary parts of the eigenvalue of the operator of linearisation, when two numbers are shown

F	P	ES	S	ν,η	$\mathbf{q}$	γ	A
6	HD	exponential	PI	0.0460	(0,1/2,1/2)	0.2593	2
6	HD	exponential	PI	0.1231	(1/2,1/2,0)	0.1049	2
6	HD	exponential	PI	0.1917	(1/2,1/2,0)	0.0401	2
9	HD	large eddies	PI	0.1046	(1/2,1/2,0)	0.1252	2
9	HD	large eddies	PI	0.2040	0	0	4
10	KD	exponential	nPI	0.0472	(0,0,1/2)	0.0711	2/2
13	KD	large eddies	nPI	0.0394	(0,1/2,0)	0.0868	2/2
14	KD	exponential	PI	0.1339	0	0	4/2
15	KD	Kolmogorov	PI	0.1491	(1/2,1/2,0)	0.0276	2
21	MHD	exponential	PI	0.1545	(0,0,1/2)	0.1353	2
22	MHD	Kolmogorov	PI	0.2483	0	0	4
22	MHD	Kolmogorov	PI	0.2710	0	0	4
23	MHD	large eddies	PI	0.0902	0	0.2618	2
23	MHD	large eddies	PI	0.2868	0	0	4

we have constructed the expansions here, is unique in that the dominant eigenvalues of the eddy diffusivity operator are imaginary at the bifurcation point. Clearly, for a parity-invariant steady state experiencing perturbation, the molecular diffusivity, for which the eddy diffusivity becomes positive, is necessarily a branching point of the branch of neutral modes for $\mathbf{q} = 0$, provided all short-scale zero-mean stability modes have negative growth rates at this critical point. The converse does not hold true: in all the 9 stability problems for parity-invariant steady states discussed *ibid.*, dominant eigenvalues of the eddy diffusivity operator pass at the branching point through zero only in the sample kinematic dynamo problem considered here. Expansions of the deviations of the growth rates, $\gamma(\eta)-\gamma(\eta_0)$, have the leading terms order ϑ^2 or ϑ^4 (the power is shown in the last column of the table); for the deviations of the imaginary parts (if applicable) of the eigenvalues of the operators of linearization, $\mathrm{Im}\lambda(\eta)-\mathrm{Im}\lambda(\eta_0) = \mathrm{O}(\vartheta^2)$. Here η_0 is the critical magnetic molecular diffusivity (in the hydrodynamic stability problem, the molecular viscosity substitutes the molecular diffusivity, and then $\vartheta = |\nu-\nu_0|^{1/2}$). The left branching in fig. 23b *ibid.* is unique in that the growth rates in the main branch also depend on molecular diffusivity, and upon branching the offshoot coexists with the main branch comprised of stability modes, which have locally, but not globally maximum growth rates (not shown in fig. 23a,b). By numerical estimations, for this branching $\gamma_m(\eta)-\gamma_o(\eta) = \mathrm{O}(\vartheta^4)$, where the subscripts m and o indicate the growth rates in the main branch and in the offshoot.

In the context of the hydrodynamic stability problem for a sample steady flow with an exponentially decaying energy spectrum (Fig. 6 *ibid.*), branch IV of stability modes of globally maximum growth rates for $\mathbf{q} = (1/2,-1/2,0)$ terminates in bifurcations of branching at both ends. A similar bifurcation is observed in Fig. 15 *ibid.* for the dynamo problem for a flow with a Kolmogorov energy spectrum, where offshoot II stems from branch I comprised of eigenmodes for $\mathbf{q} = (1/2,-1/2,0)$. The behavior looks more "canonical" in Fig. 9 *ibid.* for the stability problem for a steady flow comprised of large eddies of wave numbers 2 or less. The offshoot (branch III) of positive globally maximum growth rate modes stems from branch IV of neutral stability modes, whose zero growth rates are globally maximum. The growth rates in the offshoot and in the main branch are equal at a branching; no other conditions for these bifurcations have been identified. Both eigenvalues of the eddy viscosity operator are real and negative, unlike in the considered here magnetic dynamo.

For the linear MHD stability of parity-invariant steady states, the behavior is more involved. A similar branching was detected for all the three sample MHD steady states. For the state with an exponential energy spectrum decay (see Fig. 21 in [*Chertovskih and Zheligovsky*, 2023b]), the branching point, $\eta_0 = \nu_0$, (the computations were performed for $\nu = \eta$) is the left terminal point of branch III, where growth rates admit the global maximum for $\mathbf{q} = (0, 0, 1/2)$. The adjacent branch is I; the growth rates of its eigenmodes cease to be globally maximum inside the interval of its existence, leaving room for branch II. Small-scale instability (for $\mathbf{q} = 0$) sets in near $\nu = \eta = 0.193$ (the eigenvalues of the operator of linearization being real and strictly positive in both branches, I and III), and the combined eddy diffusivity for smaller molecular diffusivities is irrelevant.

Two instances of offshoots, stemming from branch VI comprised of neutral eigenmodes for $\mathbf{q} = 0$, are observed in Fig. 22 *ibid.* for the sample MHD steady state with the Kolmogorov energy spectrum. The dominant eigenvalues of the linearization and all the four eigenvalues of the operator of the combined eddy diffusivity are real at both branching points, the latter ones are strictly negative (unlike in the dynamo problem considered in this paper) at the right critical point, and one eigenvalue passes through zero at the left one, η_0, (i.e., the large-scale instability is present for $\eta < \eta_0$, the combined eddy diffusivity becoming negative). Near the point of bifurcation, eigenmodes in the offshoot branch V have globally maximum growth rates. Growth rates of eigenmodes constituting the offshoot branch IV are only locally maximum near the point where it stems, becoming globally maximum in interval IV at a positive distance from this point.

The arrangement is again different for the sample MHD steady state built of short-scale Fourier harmonics of wave numbers not exceeding 2 (Fig. 23 in [*Chertovskih and Zheligovsky*, 2023b]). Two offshoots stem from branches II and V comprised of short-scale eigenmodes for $\mathbf{q} = 0$. Bifurcation of branch V is similar to how the offshoot branch V stems in Fig. 22: Branch V consists now of neutral stability modes, the eigenvalues of the linearization are real, all the four eigenvalues of the operator of the combined eddy diffusivity are real and strictly negative, and the eigenmodes comprising the offshoot branch III have globally maximum growth rates near the critical points (the growth rates of the eigenmodes in the offshoot branch III lose global maximality strictly inside the interval of its existence, becoming just locally maximum and surpassed by branch IV). Unlike branch V, branch II consists of zero-mean short-scale instability modes whose growth rates are strictly positive.

Similar bifurcations of branching for non-parity-invariant states were also detected. In the kinematic dynamo problem, offshoots stem from branch I of the eigenmodes for $\mathbf{q} = (0, 0, 1/2)$ for the sample flow with an exponentially decaying energy spectrum (Fig. 10 *ibid.*) and from branch I of the magnetic modes for $\mathbf{q} = (0, -1/2, 0)$ generated by the sample flow comprised of large short-scale eddies of wave numbers not exceeding 2 (Fig. 13 *ibid.*).

We can give the following physical interpretation of the results of this paper. We have considered magnetic field generation by parity-invariant flows, which lack magnetic α-effect. On decreasing the molecular diffusivity η towards the critical point η_0 of the onset of negative magnetic eddy diffusivity, generation of large-scale magnetic modes begins, while (typically) magnetic field that has the periodicity of the flow is not yet generated (because the action of molecular diffusivity on large scales is the least). When the molecular diffusivity is close to the critical value, the most unstable generated modes constituting an offshoot branch involve very large (order $(\eta_0 - \eta)^{-1/2}$) spatial scales, which decrease fast on decreasing the molecular diffusivity below η_0. By contrast, the growth rates of the modes are very small (order $(\eta_0 - \eta)^2$) in magnitude) and they evolve very slowly. The modes are oscillatory, the frequency of oscillations behaving as $\eta_0 - \eta$.

We plan to study the asymptotics in all variants of branching encountered or suspected in this work. Moreover, it is of interest to investigate the asymptotics of co-dimension 2 branching, which may take place in the MHD stability problem on varying both ν and η independently.

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