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Doppler and Non-Doppler Shifts in Dispersion Relations for Rossby Waves and Galilean Invariance

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Abstract: The purpose of this work is to draw the reader's attention to a paradoxical fact – the existence of two different dispersion relations for linear Rossby waves: with Doppler and non-Doppler shifts. This paper highlights aspects arising from studying the interaction of Rossby waves and large-scale stationary flows within the framework of the linear wave approximation. The methods used in the work consist of the analysis of dispersion relations obtained by different authors. They are subordinated to the main task of the study – to establish where and when a non-Doppler shift appears in the system of two-dimensional linear equations of Rossby waves. Assuming that the flow is homogeneous, additional terms appear in the dispersion relation of Rossby waves for the solution in a plane wave, which can have both Doppler and non-Doppler effects. The paper shows that the non-Doppler character of the dispersion relation of Rossby waves on the current appears due to an additional assumption about the slope of the free surface, or the slope of the interface in a two-layer model (pycnocline for the ocean, and tropopause for the atmosphere). It is established that to derive some of these relations, excessive requirements for boundary conditions or separate terms in the equation for potential vorticity were previously applied. It is shown that to deduce the dispersion relation of Rossby waves with a non-Doppler shift, it is not necessary to throw out the topographic term in the boundary condition or abandon the hydrostatic approximation.

Keywords: Rossby waves, dispersion relation, doppler shift, non-doppler, Galilean invariance.

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1. Introduction

1.1. Rossby's Pioneering Work of 1939

For the simplest case of an ideal homogeneous incompressible atmosphere with purely horizontal motion, *Rossby et al.* [1939] derived a Galilean-invariant dispersion relation from the equation expressing the conservation of absolute vorticity:

$$c_1 = -\frac{\beta}{k^2} + U. \tag{1}$$

This expression was derived for one-dimensional wave perturbations of the form $v \sim \sin k(x - ct)$, where v is the meridional velocity component, x is the zonal coordinate, t is time, k is the zonal wavenumber component, c_1 is the phase velocity, independent of the meridional coordinate y. Here $\beta = \frac{\mathrm{d}f}{\mathrm{d}y}$ is the wave-generating parameter of the β-plane, characterizing the variation of the Coriolis parameter with latitude f. The x- and y-axes of the rectangular coordinate system are directed eastward and northward, respectively, $U = U_0$ is the zonal steady homogeneous flow. The second term in Equation 1, U, is commonly referred to as the "Doppler shift". A more detailed description of the Doppler shift can be found, for example, in the following sources: [*Bühler*, 2014; *Bulatov and Vladimirov*, 2015, 2017; *Erokhin and Sagdeev*, 1985a,b; *Fabrikant and Stepanyants*, 1998;

Kharif et al., 2009; *LeBlond and Mysak*, 1981; *Stepanyants and Fabrikant*, 1989; *Sutyrin*, 2020]. Sometimes, the term "quasi-Doppler shift" is used to describe these processes [*Gerkema et al.*, 2013].

In the work of *Rossby et al.* [1939], a second wave-forming parameter is also considered – the latitudinal variation in atmospheric (tropospheric) thickness. It should be recalled that the troposphere decreases in thickness from the equator to the pole: near the equator, the troposphere is approximately 12 km thick, in mid-latitudes about 10 km, and near the pole around 8 km. As a result, the tropopause, which separates the troposphere and the stratosphere, maintains a stable meridional slope. This meridional slope of the tropopause is related to the zonal stationary flow U by the geostrophic relationship

$$U = -\frac{g}{f}\frac{\partial D_0}{\partial y}, \tag{2}$$

where D_0 is the quantity on the order of the tropospheric height, U is the velocity near the base of the stratosphere. The inclination of the tropopause leads to the formation of a pressure gradient that balances the Coriolis force associated with the zonal flow [*Pedlosky*, 1987]. Figure 1 illustrates the geostrophic balance for the Northern Hemisphere. The influence of the Coriolis force in the Northern Hemisphere creates a slope of the free surface such that, for an eastward-directed flow (according to atmospheric physics terminology, such flows are called westerly, but in ocean physics, they are referred to as easterly), i.e., for $U > 0$, the layer thickness is greater at southern latitudes than at northern latitudes. This means that the total thickness of the layer in Figure 1a has a steeper slope than that determined by Equation 2. It also implies that the zonal flow for $U > 0$ enhances the β-effect (Figure 1a), meaning that for westerly zonal flow, the inclination of isopycnals leads to an intensification of the β-effect, whereas for easterly flow, it results in its weakening.

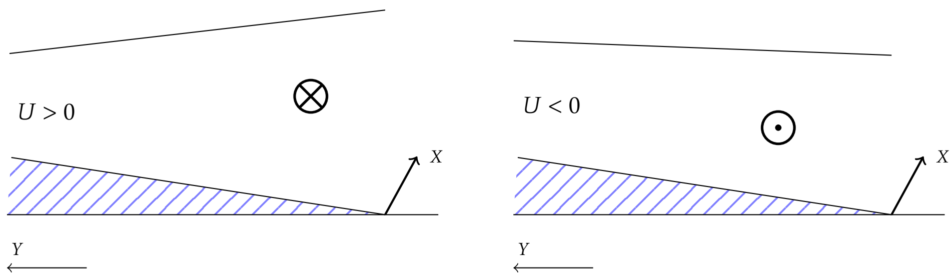

Figure 1. Free surface profile and isopycnal inclination at rest for the Northern Hemisphere ($f > 0$). The x-axis is directed eastward, and the y-axis is directed northward: (a) $U > 0$; (b) $U < 0$.

Rossby made two key assumptions that are crucial for the following discussion. Equation 2 represents the balance equation under the previously made assumptions; we will refer to it as "Rossby's first assumption".

Rossby's second assumption states that the meridional variation in the thickness of the troposphere is much smaller than the troposphere's actual thickness. As a result, the depth of the troposphere is considered a variable function and is replaced by the background flow velocity field through relation (2) when differentiated in the meridional direction. Where it is not differentiated, it can be replaced by an average tropospheric depth D_∞ (Rossby's notation). Here, we note the analogy between Rossby's assumption and the Boussinesq approximation for an incompressible fluid, as well as the β-plane approximation. Subsequently, the presence of an infinitely deep layer led to the development of so-called 1.5-layer and 2.5-layer models in modern multilayer frameworks.

Under the assumptions (1)–(2), the second one-dimensional dispersion relation was derived [*Rossby et al.*, 1939]:

$$c_2 = -\frac{\beta + FU}{k^2 + F} + U. \tag{3}$$

Since the background flow velocity U appears not only as a Doppler shift but also in the numerator of Equation 3, which is sometimes referred to as the "beta-effective" term: $\beta^* = \beta + FU$. In this case, the dispersion relation is no longer Galilean invariant. At the same time, a new term is introduced – the scale $F^{-1/2} = \sqrt{gD_\infty}/f$ which was later called the "barotropic Rossby radius". It is worth noting that in [*Rossby et al.*, 1939], the depth of the troposphere D_∞ is assumed to be 8 km. Emphasizing that (3) is also a one-dimensional dispersion relation, where the denominator contains not the full squared wavenumber but only its zonal component k.

Thus, within the framework of a homogeneous single-layer model, *Rossby et al.* [1939] derived two distinct one-dimensional dispersion relations. The first (see Equation 1) contains only one wave-forming parameter – the latitudinal variation of the Earth's rotation frequency, i.e., the β-parameter. Equation 1 is Galilean-invariant since the background flow velocity appears only as a Doppler shift. The second dispersion relation (see Equation 3) involves two wave-forming parameters: in addition to the β-parameter, there is also the meridional variation in the height of the troposphere. As a result, Equation 3 is not Galilean-invariant, as the flow velocity U appears not only as a Doppler shift but also in the so-called "β-effective" term.

It should be noted that Rossby himself did not use the terms "Doppler" and "non-Doppler" shift. In his terminology, the dispersion relation (1) is referred to as "approximate," while relation (3) is called "exact".

1.2. Further Research Following Rossby's 1939 Pioneering Work

The one-dimensional dispersion relation (1) was generalized by *Haurwitz* [1940] to a two-dimensional case, incorporating a finite transverse wavenumber:

$$c_3 = -\frac{\beta}{k^2 + l^2} + U. \tag{4}$$

Here, k is the zonal component and l is the meridional component of the wavenumber.

The generalization of the dispersion relation (3) to the two-dimensional case took more time than for the one-dimensional case and was apparently first achieved in the works of *Charney* [1947, 1948]. The generalization of relation (3) to the two-dimensional case can be obtained by expressing the solution in the form of a classical double Fourier integral $\sim \exp(+i(kx + ly - \omega t))$ and substituting it into *Charney* [1948, Eq. (58a)]. As a result, we obtain the generalization of the dispersion relation (3) for the two-dimensional case:

$$c_4 = \frac{\beta + FU}{k^2 + l^2 + F} + U. \tag{5}$$

This result is usually attributed to *Pedlosky* [1987, Eq. (3.18.9)] and *Vallis* [2017, Eq. (6.65)], although in fact, it was obtained much earlier.

Later, the term "*non-Doppler effect*" first appeared in the works [*Corby*, 1967; *Lindzen*, 1968], where the authors pointed to the lower boundary condition as the cause of this phenomenon: "an important non-Doppler effect enters into the lower boundary condition. The lower boundary condition determines the free oscillations' equivalent depths which now depend on the horizontal wave numbers and can be negative – consistent with Corby's results" [*Lindzen*, 1968]. Later, *White* [1977] generalized the results of *Charney* [1947, 1948] for a stratified medium, abandoning the hydrostatic approximation, and proposed an explanation for the non-Doppler shift as "a preferred frame of reference".

Subsequently, in the literature, the term "the non-Doppler effect" became associated with the dispersion relation (5) [see, e.g., *Dewar*, 1998; *Held*, 1983; *Killworth and Blundell*, 2005, 2007; *Killworth et al.*, 1997; *Kubokawa and Nagakura*, 2002; *Liu*, 1999a,b; *Maharaj et al.*, 2007; *Samelson*, 2010; *Tulloch et al.*, 2009; *Verdiére and Tailleux*, 2005].

In the work [*Kravtsov and Reznik*, 2020], the term "*Galilean non-invariance*" is used to describe the non-Doppler shift. In [*Nycander*, 1994], the same phenomenon is referred to as a "counterintuitive result". In [*Gulliver and Radko*, 2022], for a 2.5-layer model, this effect is called "*anti-Doppler*". In [*Morel*, 1995; *Morel and McWilliams*, 1997; *Yasuda et al.*, 2000], the term "*pseudo-β effect*" is used.

It is important to note that all the authors mentioned above, who introduce new and new terms, unfortunately, do not reference Rossby's original work [*Rossby et al.*, 1939].

When reading the most well-known monographs that present the dispersion relation of Rossby waves on a current (see, for example, [*Pedlosky*, 1987; *Vallis*, 2017]), one might get the impression that if the Rossby radius is finite: $F \neq 0$, then the dispersion relation (5) is Galilean non-invariant, i.e., it exhibits a non-Doppler shift. Conversely, if the Rossby radius is infinite: $F = 0$, meaning that non-divergent waves are considered, then we obtain the classical Doppler shift, and Equation 4 is Galilean invariant. This is exactly how *Vallis* [2017] presents this aspect in his monograph.

Note that Equation 4 can be derived from Equation 5 by setting a specific condition. However, the peculiarity of the situation lies in the fact that, under this condition, there exists another dispersion relation, which we will derive below, that remains Galilean invariant. This dispersion relation is presented in the monograph [see *Kundu et al.*, 2015, Eq. (13.120)] but there it is given without proof. In contrast, *Bühler* [2014] provides a derivation of this equation [see *Bühler*, 2014, Eq. (9.19); *Korotaev*, 1988, Eq. (1.1.9)]. At the same time, in the monograph [*LeBlond and Mysak*, 1981] the possibility of a non-Doppler shift is fundamentally not considered, and the Doppler shift is assumed to be an unshakable property of all types of linear waves.

1.3. Objective of the Study

In this study, we derive and compare two dispersion relations in a linear barotropic formulation: (4) with a Doppler shift and (5) with a non-Doppler shift. Furthermore, we demonstrate how to transform the shallow water equations to introduce a non-Doppler term without the need for a laborious scale analysis. Thus, the objective of this study is to draw the reader's attention to a paradoxical fact – the existence of two different dispersion relations within the same formulation: one with a Doppler shift and one with a non-Doppler shift. This objective is achieved by solving the following problem: determining where and under what conditions a non-Doppler shift appears in the system of two-dimensional linear equations for Rossby waves.

It is important to note the following: the first dispersion relation with a Doppler shift is derived within the framework of a single-layer barotropic model, whereas the second equation with a non-Doppler shift, when $F \neq 0$, inherently contains baroclinic elements, as it is derived from a two-layer baroclinic model. Its significance for Rossby waves has been emphasized by many authors, including [*Killworth and Blundell*, 2005, 2007; *LaCasce*, 2017; *Maharaj et al.*, 2007; *Nezlin*, 1986; *Schlax and Chelton*, 2008; *Tailleux and McWilliams*, 2001; *Tulloch et al.*, 2009; *Wunsch*, 2015].

It is important to note that the existence of two dispersion relations is not reflected in any known monograph. Existing monographs typically consider the shallow water model and the low-frequency approximation (small Rossby number), where the quasigeostrophic approximation is automatically assumed, leading to the derivation of dispersion relation (5). However, there is no mention of the existence of the second dispersion relation, the derivation of which we will present below.

2. Analysis of the Basic Equations

Let us consider the shallow water approximation on a β-plane in the presence of a free surface and topographic variations. Within this approach, we will generalize the results of *Lounguet-Higgins* [1965] and *LeBlond and Mysak* [see also 1981, Eqs. (15.50–15.51)], to the case of a zonal stationary flow. The linearized shallow water equations for a rotating fluid on the background of a uniform current in the x-direction $\overrightarrow{U} = (U, 0)$ and one-dimensional bottom topography variations, which, for simplicity, vary only in the y-direction, are given as follows:

$$u_1 + U u_x - f v = -g\eta_x, \tag{6}$$

$$v_t + Uv_x + fu = -g\eta_y, \tag{7}$$

$$(Hu)_x + (Hv_y) + \eta_t = 0, \tag{8}$$

This decomposition is valid for small Rossby numbers, $R_0 \ll 1$ (the ratio of nonlinear terms to the Coriolis force). Here, $\eta = \eta(x, y, t)$ represents the deviation of the free surface from its equilibrium position, and $H = H(y)$ is the meridionally varying bottom topography (with the assumption that $H \gg \eta$).

The flow is stationary in time and uniform in the zonal (longitudinal) direction. The solution of the system can be sought in the form of a Fourier integral. We assume:

$$u, v, \eta \sim \exp i(kx - \omega t). \tag{9}$$

Then, equations (6)–(8), under condition (9), take the following form:

$$i(kU - \omega)u - fv = -gik\eta, \tag{10}$$

$$i(kU - \omega)v - fu = -g\eta_y, \tag{11}$$

$$ikhu + (hv)_y + i(kU - \omega)\eta = 0. \tag{12}$$

The system of equations (10)–(12) is Galilean invariant. No non-Doppler effect is initially embedded in these equations, and this is a crucial point to emphasize.

Next, through straightforward transformations from (10) and (11), we obtain:

$$\left[f^2 - (kU - \omega)^2\right]u = g\left[k(kU - \omega)\eta - f\eta_y\right], \tag{13}$$

$$\left[f^2 - (kU - \omega)^2\right]v = ig\left[fk\eta - (kU - \omega)\eta_y\right]. \tag{14}$$

Let us introduce the notation $\omega_d = \omega - kU$. Next, (13) and (14) should be substituted into (12) [*Lounguet-Higgins*, 1965].

2.1. Free Surface and Consideration of Topography on the β-Plane

The combined consideration of the β-plane, topography, and free surface leads to the following differential equation:

$$\left(h\eta_y\right)_y + \left(\frac{\omega_d^2 - f^2}{g} - hk^2 + \frac{fk}{\omega_d}h_y + \frac{\beta h}{\omega_d}\left[k - \frac{2f^2k}{f^2 - \omega_d^2}\right]\right)\eta - \frac{2f\beta h}{f^2 - \omega_d^2}\eta_y = 0. \tag{15}$$

Since $h = h(y)$, no dispersion relation can be derived from (15).

Let us make an additional approximation: $\omega_d^2 \ll f^2$ (low-frequency approximation) and discard the last term in Equation 15. This can be done because, in the β-plane approximation, the last term is small compared to, for example, the first term, where $h_y/h \sim L_y^{-1}$, with L_y being the scale of topographic variation in the meridional direction, while $2\beta/f \sim R^{-1}$, where R is the Earth's radius. Under these assumptions, we obtain:

$$\left(h\eta_y\right)_y - h\left(\frac{f^2}{gh} + k^2 + \frac{k}{\omega_d}\left[\beta - f\frac{h_y}{h}\right]\right)\eta = 0. \tag{16}$$

Equation 16 includes the topographic derivative and the free surface as physical variables. It represents a generalization of equation (20.13b) from the monograph *LeBlond and Mysak* [1981, Eq. (20.13b)] (for the case of meridional topography).

2.2. β-plane Approximation, Free Surface, Bottom Topography Neglected

When bottom topography variations are not taken into account, i.e., $h = H_0 = \text{const}$, then for $\eta \ll H_0$, Equation 16 takes the form:

$$\eta_{yy} - \left(\frac{f^2}{gH_0} + k^2 + \frac{\beta k}{\omega_d} \right) \eta = 0. \tag{17}$$

Since all the coefficients in Equation 17 are constant under the adopted approximation (i.e., there is no dependence on the transverse coordinate y), we obtain the dispersion relation for barotropic Rossby waves on a current, taking into account the free surface and in the absence of bottom topography variations:

$$\omega_1 = -\frac{\beta k}{k^2 + l^2 + F} + kU. \tag{18}$$

Here $F = f^2/gH_0$. This dispersion relation has the classical Doppler shift and is Galilean invariant.

2.3. Non-Doppler Shift

Let's perform a change of variables $\eta(y) = h(y)^{-1/2}\psi(y)$. Then Equation 16 takes the form:

$$\psi_{yy} + \left(-\frac{1}{2h^{1/2}} \left[\frac{h_y}{h^{1/2}} \right]_y - \frac{f^2}{gh} - k^2 - \frac{k}{\omega_d} \left[\beta - f\frac{h_y}{h} \right] \right) \psi = 0. \tag{19}$$

Since $h = h(y)$, additional assumptions need to be made to derive the dispersion relation in order to obtain an equation with constant coefficients.

Let's apply the Rossby approximations. In the last term of the numerator, we use the equality $h_y = -fU/g$ and assume the bottom topography to be a constant where it is not differentiated with respect to the meridional coordinate. Formally, this can be expressed by the formula $h = h_0 + \varepsilon h(y)$, where ε is a small parameter (the Rossby number). Thus, the first term in Equation 19 can be discarded, considering it to be of second-order smallness. As a result, we again obtain the dispersion relation (5). The second derivative gives the meridional wavenumber l^2:

$$\omega_2 = -\frac{(\beta + FU)k}{k^2 + l^2 + F} + kU. \tag{20}$$

We emphasize in this study that the original linear shallow water equations, or long-wave equations, are Galilean invariant, and from them, the dispersion relation (18) with the Doppler shift is derived. However, obtaining the non-Doppler relation (20) requires an additional assumption, which we refer to as the "Rossby approximation".

In the works of *Charney* [1947, 1948], as well as in subsequent monographs [*Pedlosky*, 1987; *Vallis*, 2017], the dispersion relation (5) is obtained through a laborious analysis of scales and asymptotic expansions in terms of a small parameter. However, in these studies, the authors effectively use the Rossby approximation, which may lead the reader to mistakenly believe that the shallow water approximation itself is responsible for the non-Doppler shift. However, this is not the case. The true cause of the non-Doppler shift is the baroclinicity of the background flow.

2.4. Exponential Topography Profile

There is an exception when it is possible to derive a dispersion relation for variable bottom topography by applying the "rigid lid" approximation. In this case, an exponential topography profile is used: $h(y) = H_0 \exp(-\alpha y)$ where the parameter α, characterizing the slope profile (convex or concave), can be either positive or negative.

The dispersion relation in this approach is obtained differently, using the mass stream function. This approach is discussed in more detail in the monograph [*LeBlond and Mysak*, 1981], as well as in the works of [*Gnevyshev et al.*, 2023a,b]. For an exponential topography profile, we obtain the following dispersion relation:

$$\omega_0 = -\frac{(\beta + \alpha f)k}{k^2 + l^2 + \frac{1}{4}\alpha^2} + kU, \quad h(y) = H_0 \exp(\alpha y). \tag{21}$$

The analysis of topographic waves is highly diverse. Depending on the characteristics of the bottom topography, various types of waves are distinguished, including topographic shelf waves, internal shelf waves, transverse waves, trench waves, Kelvin waves, and double Kelvin waves as limiting cases of stepped topography [*Churilov and Stepanyants*, 2022; *Efimov et al.*, 1985; *Gnevyshev et al.*, 2023a,b; *Rabinovich*, 1993]. The role of the β parameter here is played by the so-called "effective beta", where the contribution of the topographic component is compared with β. For significant bottom slopes, the topographic component may dominate. Along continental slopes, and especially in oceanic trenches, the topographic factor becomes the dominant influence [*Drivdal et al.*, 2016; *Gnevyshev et al.*, 2021a, 2019, 2023a,b; *Gnevyshev et al.*, 2022].

3. Discussion and Conclusions

Thus, the analysis of dispersion relations leads to the following conclusions:

1. The original system of shallow water equations, linearized against the background of a zonal stationary flow (10)–(12), is Galilean invariant. We want to emphasize the following paradoxical fact, which is often overlooked: this initially Galilean invariant system (let us stress this) has two dispersion relations. The first dispersion relation (18) includes the classical Doppler shift, which is entirely logical and expected since the original system is Galilean invariant. However, the second dispersion relation (20) is no longer Galilean invariant, which is where the paradoxical nature of the situation manifests itself.
2. The reason for this non-invariance is most concisely formulated in the work of *Tulloch et al.* [2009] – we allow ourselves to provide a quote: "Mean currents change the structure of the waves in two ways: by altering the background potential vorticity gradient (sometimes so much so that β is negligible), and by Doppler shifting the signal. A number of authors [*Dewar and Morris*, 2000; *Killworth and Blundell*, 2007; *Killworth et al.*, 1997; *Maharaj et al.*, 2007] have shown that the straightforward inclusion of the mean thermal wind currents in the linear Rossby wave problem leads to a much closer agreement between the observed phase speeds and theory". This phenomenon is also partially discussed in the article by *Nezlin* [1986]:"It is interesting to note that for very long Rossby waves, the influence of wind on their propagation speed disappears: the advection of the wave by the wind is exactly compensated by an increase in the wave speed relative to the wind due to the induced hydrostatic pressure gradient".
3. In our work, we draw the reader's attention to the fact that, for the one-dimensional case, both of these dispersion relations were already derived in Rossby's pioneering work of 1939, but only as a consequence under certain additional assumptions.
4. This naturally raises the question: at what point does the non-invariance emerge in the Galilean-invariant system (10)–(12)? To answer this question, let us refer to Equation 12 and examine its second term $(hv)_y$. We rewrite it as the sum of two terms: $(hv)_y = h_y v + h v_y$. Next, following *Rossby et al.* [1939], we make the first assumption (approximation) that the velocity field of the background flow U is related to the topography h by the relation $h_y = -Uf/g$ (this corresponds to Equation 2 in Rossby's pioneering work). Then, the first term transforms as follows: $h_y v = -vUf/g$. Further, we replace $h(y)$ with a constant H_0 wherever it is not being differentiated. As a result, we find that it is precisely these two assumptions that render the dispersion relation (20) Galilean non-invariant. This is the key point we aim to highlight for the reader.

5. It is also important to consider the following remark. In the fundamental monographs [*Pedlosky*, 1987; *Vallis*, 2017], the sections dedicated to the analysis of the shallow water equations do not include the dispersion relation (18) with the classical Doppler shift. As a result, upon encountering the Galilean non-invariant relation (20), the reader might mistakenly conclude that the shallow water equation system itself is not Galilean invariant. However, this is not the case. The reason is that in these monographs [*Pedlosky*, 1987; *Vallis*, 2017], the Rossby approximation is already implicitly embedded within the scaling analysis. Consequently, it becomes impossible to derive the Galilean invariant dispersion relation (18) from the analyzed system of equations.
6. On the other hand, in Kundu's monograph [*Kundu et al.*, 2015, Eq. (18)], is presented, albeit without derivation [*Kundu et al.*, 2015, see p. 756, Eq. (13.120)]. However, there is no mention of the Galilean non-invariant relation (20). In this monograph, as well as in [*LeBlond and Mysak*, 1981], only one type of wave-current interaction is considered – the Doppler shift – without any discussion of the existence of a non-Doppler shift. The derivation of relation (18) can also be found in [*Bühler*, 2014, Eq. (9.19), p. 194]. It is also worth noting that *Pedlosky* [1987, Sec. 3.18] and Bühler *Bühler* [2014] start from the same vorticity equation, yet they arrive at fundamentally different results: Pedlosky derives relation (20), while Bühler obtains relation (18).
7. In this publication, the authors aim to draw the reader's attention to a remarkable and paradoxical fact – the possibility of obtaining a Galilean non-invariant dispersion relation from an initially Galilean-invariant system of equations. We demonstrate at which stage the transition to non-invariance occurs and how different authors approach this issue in various ways. The reason that allows such unusual results in geophysical hydrodynamics, especially in the long-wave approximation, is vividly and accurately described by *Gill* [1982, Sec. 7.4]: "Quasi-geostrophic motions are rather delicate processes because the equations are not exactly satisfied". This provides some insight into the paradox mentioned above.
8. We hope that our analysis will enable readers to view certain well-known facts from a new perspective. The novelty of this study lies in identifying and explaining the paradoxical loss of Galilean invariance in an initially invariant system, determining the exact moment of transition to non-invariance, and critically analyzing and systematizing approaches to this problem in the scientific literature. In other words, we have demonstrated that even in an initially invariant system, non-invariant properties can emerge under certain conditions – an aspect previously overlooked – and we have pinpointed the moment when Galilean invariance is lost. We attribute the emergence of a Galilean non-invariant relation within an invariant system to the delicate nature of quasi-geostrophic processes. The equations governing such processes are not strictly satisfied, creating conditions for unexpected and paradoxical results, particularly in the long-wave approximation.
9. In modern oceanography, a persistent trend has emerged where the analysis of Rossby wave propagation – a fundamental type of planetary wave playing a pivotal role in the dynamics of large-scale oceanic processes – is increasingly conducted indirectly, by observing the trajectories and movement of coherent mesoscale eddies. This approach stems from the understanding that eddies, possessing the capacity for prolonged existence and displacement over significant distances, can serve as distinct "indicators" or "carriers" of Rossby wave signals. Their observed trajectories, particularly their zonal and meridional displacements, frequently correlate with the theoretically predicted phase or group velocities of these waves. This methodology allows researchers to investigate the complex dynamics of planetary waves even under conditions of limited direct measurements or when traditional methods for detecting Rossby waves are hampered by the superposition of other dynamic processes. Consequently, mesoscale eddies have become an invaluable tool for decoding large-scale oceanic movements and elucidating their intricate relationship with wave phenomena [*Belonenko et al.*,

2018; *Belonenko and Frolova*, 2019; *Belonenko et al.*, 2016; *Belonenko and Kubryakov*, 2014; *Fedorov and Belonenko*, 2020; *Gnevyshev and Belonenko*, 2023; *Gnevyshev et al.*, 2019, 2021b].

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