Explicit asymptotics for tsunami waves in framework of the piston model

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[1] An asymptotically numerical description of tsunami waves propagation in a basin with non-uniform depth in a neighborhood of wavefronts that can have caustics is proposed. The piston model and the long wave approximation are used. It is assumed that the size of the area of the initial disturbance is small in comparison both with the characteristic length of interval of the varying of the bottom depth and the distance from the observation point. The description is based on a generalization of asymptotic approach known as the Maslov canonical operator. We find formulas that are relatively simple and can be transformed in a computer program for fast calculating wave profiles. Some features of the tsunami waves propagation in basins of non-uniform depth are illustrated by graphics. *INDEX TERMS:* 4564 Oceanography: Physical: Tsunamis and storm surges; 4594 Oceanography: Physical: Instruments and techniques; 4599 Oceanography: Physical: General or miscellaneous; *KEYWORDS:* tsunami, asymptotics, dispersion-less piston model, wavefront, wave-profile.

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1. Introduction

[2] The traditional modelling of the tsunami waves propagation in the open ocean is done by solving the linear hydrodynamical equation in 2-D long wave approximation and in the framework of the so called "piston model", which assumes that the source of the tsunami is given by an instantaneous vertical velocity of a certain region of the bottom of the ocean which generates an uplift of the ocean surface

$$\frac{\partial^2\eta}{\partial t^2} = \langle\nabla, C^2(x)\nabla\rangle\eta, \quad C(x) = \sqrt{gH(x)},$$
$$x = (x_1, x_2) \in \mathrm{R}_x^2, \tag{1}$$
$$\eta|_{t=0} = V(\frac{x}{l}), \quad \eta_t|_{t=0} = 0. \tag{2}$$

Here $\eta(x,t)$ is the elevation of the ocean surface, $H(x)$ is the depth of the basin, g is the gravity acceleration, and $V(x/l)$ is the uplift of the ocean surface localized in the area of a characteristic size l. It is assumed that l is small in comparison both with the characteristic length of the interval of change of the bottom depth and the distance from the observation point. This means in particular also that the function $V(y)$ decays fast as $|y| \to \infty$. Usually the problem (1, 2) is solved numerically with finite difference methods. However analytical formulas of solutions are useful from different points of view. The main reason is that it is not so easy to use direct numerical methods for real time tsunami warning, because they take too much time and require too much information on the tsunami sea bottom source.

[3] Problem (1, 2) seems like a classical one for mathematical physics and asymptotical analysis. Nevertheless the explicit formula for its solution (which can be transformed in a computer program for fast calculation of wave profiles) were obtained quite recently, although some asymptotic representation was given in [*Dobrokhotov et al.,* 1991]. The main mathematical difficulties here are related with the metamorphosis of the solution: it is localized in the neighborhood of the point $x = 0$ (the origin) at $t = 0$, but after some

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time it changes its structure taking the form of a function localized in the neighborhood of a closed curve (the wave front), which in turn can have sometimes self-intersection and singular (focal) points. This phenomena was described in asymptotic theories for fast oscillating and non smooth solutions of a wide range of partial differential equations. The global representation for fast oscillating solutions (with effects of focalization taken into account) is given by the Maslov canonical operator [*Maslov,* 1965; *Dobrokhotov and Zhevandrov,* 2003]. However, one cannot apply this theory to problem (1, 2) directly because the solutions in this case have a different structure. Nevertheless two simple ideas allow one to modify the Maslov approach and obtain explicit asymptotic formulas for the solutions of (1, 2): 1) the problem about localized solutions can be transformed to the one about fast oscillating solutions with a Fourier-type integral transform, 2) the final formulas can be simplified if one takes into account the ideas from boundary layer expansions near the wave fronts. We combine these ideas together with the Maslov theory [*Maslov,* 1965; *Maslov and Fedoiuk,* 1981; *Dobrokhotov and Zhevandrov,* 2003] and results from [*Dobrokhotov et al.,* 1991; *Maslov and Fedoiuk,* 1989]. Finally we propose an asymptotic-numerical description of tsunami in a basin with non-uniform depth in a neighborhood of wavefronts that can have caustics. This approach takes into account in a simple and direct way physical effects (like the metamorphosis of the tsunami front mentioned above) coming from the singularities related with the Hamiltonian system i.e. focal points and caustics. The presented formulas can be transformed in a computer program by means of the software of the type of Mathematica or Maple for calculating wave profiles and so they can be used for a reliable early warning system. Here we explain the meaning of the final formulas announced by *Dobrokhotov et al.,* [2006a] (some more details can be found in [*Dobrokhotov et al.,* 2006b]). In the graphics we show that many features of the tsunami wave propagation in such basins can be explained by means of straightforward formulas without any additional complications.

2. General Equations and Asymptotic Formula for Wave Profile in the Uniform Depth Basin

2.1. The General Equations

[4] Let us give first very rough arguments showing the possibility to use (1, 2) for a description of tsunami waves in a frame of the piston model over slow varying bottom. The equations (1) are obtained from the following linear equations of the piston model without the assumption of long wave approximation for the velocity potential $\Phi(x, z, t)$ and the elevation of the free surface $\eta(x, t)$ (see e.g. [*Whitmore and Sokolowski,* 1996; *Pelinovski,* 1996; *Kowalik and Murty,* 1993; *Lewis and Adams,* 1983; *Shokin et al.,* 1989; *Tinti,* 1993])

$$\begin{gathered}\Delta\Phi = 0, \quad x = (x_1, x_2) \in R^2, \quad -H(x) < z < 0,\\ \eta_{\,t} - \Phi_z\,|_{z=0} = 0, \qquad \Phi_t\,|_{z=0} + g\eta = 0,\\ \Big[\Phi_z + <\nabla H, \nabla\Phi>\Big]\,|_{z=-H(x)} = V(\frac{x}{l})\delta(t)\end{gathered} \tag{3}$$

where δ is the delta function, and it is assumed that $\Phi \equiv 0$, $\eta \equiv 0$ when $t < 0$.

[5] The task is to evaluate $\eta(x, t)$ at large distances $|x| >> l$ from the source, near the wavefronts (where $|\eta|$ has its maximum values), assuming that $l >> H(x)$ and that the depth $H(x)$ has a small variation at distances of the order l.

2.2. Asymptotic Formula

[6] In the case of the basin of a uniform depth H_0, the wavefronts are the circles $|x| = C_0 t$, where $C_0 = \sqrt{gH_0}$ is the velocity of the long waves. We seek the solution for the problem (3) in the form of the Fourier transform with respect to $x = (x_1, x_2)$ and with Fourier parameter $p = (p_1, p_2)$. Using in the integrand polar coordinates (ρ, ϕ) defined by the formulas $p_1 = \rho/l\cos\phi, p_2 = \rho/l\sin\phi$ we obtain

$$\begin{gathered}\eta(x,t) =\\ \tfrac{1}{2\pi}\int_0^\infty \frac{\rho}{\cosh\frac{H_0\,\rho}{l}}\cos\Big(\frac{\rho\, C_0 t}{l}\sqrt{\frac{l}{H_0\,\rho}\tanh\frac{H_0\,\rho}{l}}\Big)d\rho\,\times\\ \int_0^{2\pi}\widetilde{V}(\rho,\phi)\exp\big[i\tfrac{|x|}{l}\rho\cos(\phi-\psi)\big]d\phi,\end{gathered} \tag{4}$$

where

$$\begin{gathered}\widetilde{V}(\rho,\phi) = \frac{1}{2\pi}\int_{\mathrm{R}^2_y} V(y)e^{-i\rho\ \mathbf{n}\ y}dy,\\ \mathbf{n}(\phi) = (\cos\phi, \sin\phi)^{\mathrm{T}}, \quad y = \frac{x}{l}\end{gathered} \tag{5}$$

and the angle ψ is defined by the equations

$$x_1 = |x|\cos\psi,\ x_2 = |x|\sin\psi.$$

Below, we consider in detail the case, when the initial vertical displacement of the bottom is like a ridge (or a valley), though the approach developed here is applicable for any other localized disturbance.

[7] To make final formulas more explicit let us model the initial displacement using the function of the type of the Gaussian exponent with oscillations:

$$\begin{gathered}V(y) = V_0\cos(a_1 Y_1 + a_2 Y_2 + \chi)e^{-b_1 Y_1^2 - b_2 Y_2^2},\\ Y = \Theta(\theta)y, \quad \Theta(\theta) = \begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix},\end{gathered} \tag{6}$$

where $a_1, a_2, 1 < b_1, b_2 < 3, \theta, \chi$ are dimensionless parameters and V_0 is the parameter of the dimension of length.

[8] Substituting for $V(y)$ from (6) in (5) we obtain

$$\widetilde{V}(\rho,\phi) = \frac{V_0}{2\sqrt{b_1 b_2}}e^{-\sigma-\beta\rho^2}\cosh(\gamma\rho + i\chi), \tag{7}$$

where

$$\begin{gathered}\sigma = \frac{1}{4b_1 b_2}(b_1 a_2^2 + b_2 a_1^2),\\ \beta = \frac{1}{4b_1 b_2}\big[b_1\sin^2(\phi-\theta) + b_2\,\cos^2(\phi-\theta)\big],\\ \gamma = \frac{1}{2b_1 b_2}\big[b_1 a_2\,\sin(\phi-\theta) + b_2 a_1\,\cos(\phi-\theta)\big].\end{gathered} \tag{8}$$

Now we derive the equation (9) under the main assumption $|x| >> 1$. We estimate the integral with respect to θ by the stationary phase method. After some calculations we have the following asymptotic formula for $\eta(x,t)$ (see [*Borovikov and Kelbert,* 1996; *Dobrokhotov et al.,* 1993; *Berry,* 2005]):

$$\eta(x,t) \approx \frac{1}{\sqrt{2\pi}}\sqrt{\frac{l}{|x|}} \; * \\ \mathrm{Re}\Big[e^{-i\pi/4}\int_0^\infty \frac{\widetilde{V}(\rho,\psi)\sqrt{\rho}}{\cosh(\frac{\rho H_0}{l})}\exp\big[i\,\rho\,\Big(\frac{|x|}{l} - \\ \frac{C_0 t}{l}\sqrt{\frac{l}{\rho H_0}\tanh(\frac{\rho H_0}{l})}\Big)\big]d\rho\Big]. \tag{9}$$

[9] From (7) and (8) one can conclude, reminding the above assumption on the values of the parameter b, that the main contribution to the integral (9) corresponds to the values of ρ in the interval $0<\rho<3$. Moreover, since $\frac{|x|}{l} >> 1$ integral (9) is not small only in the case when the expression in the big parentheses in the exponent of the integrand is small. Further, since $\frac{H_0}{l} << 1$, the functions $\frac{1}{\cosh(\rho\frac{H_0}{l})}$ and $\sqrt{\frac{l}{\rho H_0}\tanh(\frac{\rho H_0}{l})}$ in the integrand can be expanded in Taylor series in the powers of $\frac{\rho H_0}{l}$:

$$\frac{1}{\cosh(\rho\frac{H_0}{l})} = 1 - (\frac{\rho H_0}{l})^2 + \mathrm{O}(\frac{\rho H_0}{l})^4, \\ \sqrt{\frac{l}{\rho H_0}\tanh(\frac{\rho H_0}{l})} = 1 - \frac{1}{6}(\frac{\rho H_0}{l})^2 + \mathrm{O}(\frac{\rho H_0}{l})^4 . \tag{10}$$

One can see that the integral (9) gets its largest values near the circle $|x| = C_0 t$, i.e. in a neighborhood of the wavefront. Then, if we take in series (10) the first terms only, we obtain from (9) the formulae for $\eta(x,t)$ in the long waves approximation. From (9) and (10) we can derive roughly the following requirement for the correctness of the long wave approximation

$$|x| << \frac{l^3}{H_0^2}. \tag{11}$$

Taking $H_0 = 4$ km, $l = 50$ km we find from (11) that for the typical conditions of the ocean, equation (1) can be used instead of the general equations (3) in the piston model in the case of a basin of uniform depth. We note that the same conclusion can be done for initial disturbances of more general types than those described by (6) as well as in the case of a basin with a non-uniform slowly varying depth. According to this reason we will consider here the problem in the long wave approximation (1, 2) instead of the general equations (3).

[10] Thus, from (9) an asymptotic formula for $\eta(x,t)$ at any instant t and observation point x located near wavefront and satisfying requirement $|x| >> l$ and (11) becomes

$$\eta(x,t) \approx \sqrt{\frac{l}{|x|}}\;\mathrm{Re}\,F\Big(\frac{\Phi(x,t)}{l},\psi(x)\Big), \\ \Phi(x,t) = |x| - C_0 t, \tag{12}$$

where ψ is the angle between the vector x and the axis Ox_1,

$$F(z,\psi) = \frac{e^{-i\pi/4}}{\sqrt{2\pi}}\int_0^\infty f(\rho,\psi)e^{iz\rho}d\rho, \\ f(\rho,\psi) = \sqrt{\rho}\widetilde{V}(\rho,\psi). \tag{13}$$

In (13) the function $\widetilde{V}(\rho,\psi)$ is given by (5) for an initial disturbance of a general type and by (7) for the disturbance (6) considered here. In the last case the integral in (13) is evaluated analytically and $F(z,\psi)$ becomes

$$F(z,\psi) = \frac{V_0}{\sqrt{32\pi b_1 b_2}}[(Q_+ + Q_-)], \\ Q_\pm(z,\psi) = \\ \frac{e^{-[\,\sigma + i(\frac{\pi}{4}\mp\chi)]}}{8\,\beta^{\,5/4}}\Big(-\sqrt{\beta}\,\Gamma(-\frac{1}{4})\ \ {}_1F_1(\frac{3}{4},\,\frac{1}{2},\,\frac{w_\pm^2}{4\beta}) + \\ w_\pm\,\Gamma(\frac{1}{4})\ \ {}_1F_1(\frac{5}{4},\,\frac{3}{2},\,\frac{w_\pm^2}{4\beta})\Big), \tag{14}$$

where

$$w_\pm = \ \pm\gamma + iz$$

and ${}_1F_1(...)$ are the Kummer's confluent hypergeometric functions. The function F can be expressed also in terms of parabolic cylinder functions D_ν (see [*Dobrokhotov et al.,* 1991]), we use here the form given by Mathematica.

[11] One can see from (12) that the function $F(z,\psi)$ defines the structure of wave profiles near the wave front (i.e. for values of z that are small or are of the order of several units). In turn, the function $F(z,\psi)$ depends on the form of the initial bottom disturbance (through function $\widetilde{V}$ in (13)) and the variable z. The dependence $\mathrm{Re}F(z,\psi)$ on z is presented in Figures 1 and 2 for different values of the angle ψ.

[12] Figure 1 shows that in the case $b_1 \neq b_2$ (non axially symmetric initial bottom disturbance) the wave profile has different forms at different values of the angle ψ, while Figure 2 shows that the difference becomes very strong if $a_1 \neq a_2$. So the wave profile crucially depend on the form of initial disturbance determined by (6).

[13] In the case of the initial disturbance of general type, the function $F(z,\psi)$ can be calculated from (13) and (5) numerically. The calculation is simplified by the fact that the important part of the integral (13) corresponds to the values of the variable of integration ρ of the order of several units.

[14] Keeping in mind that the structures of the asymptotic formulae for $\eta(x,t)$ are similar in the cases of the basins of uniform and non uniform depth (see Sec 3), we give the following comments on the formulae (12) using terminology of the WKB theory.

[15] The function $\Phi(x,t)$ in (12) can be called the phase because of two facts. First, from (12) and (13) it is clear that $|\eta(x,t)|$ has a maximum when $\Phi(x,t) = 0$ and decays rapidly with increasing $|\Phi(x,t)|$. Second, outside of the neighborhood of origin $x = 0$ the function $\Phi(x,t)$ is the action or a solution of 2-D Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + C_0|\nabla S| = 0 \tag{15}$$

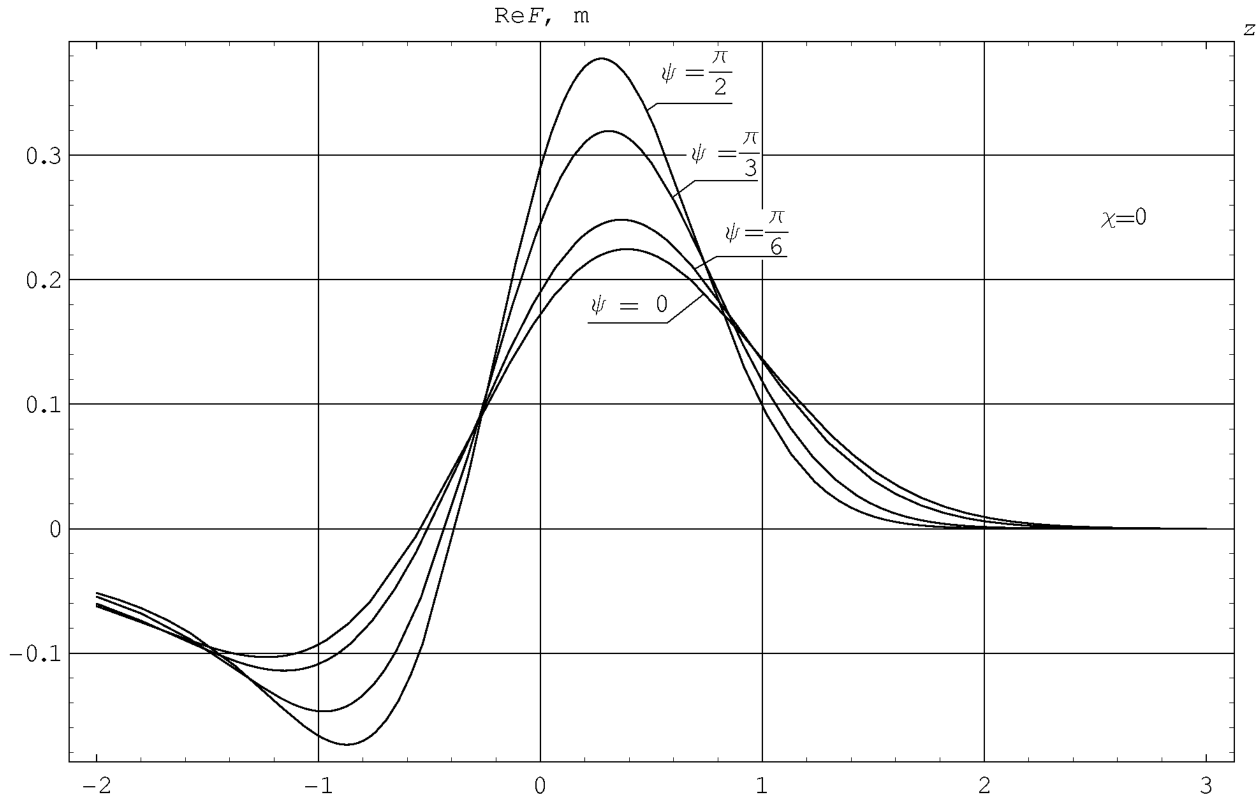

Figure 1. Function $\mathrm{Re}F(z, \psi)$ for $V(y)$ determined by (6) where $V_0 = 1\,\mathrm{m}$, $a_1 = a_2 = 0$, $b_1 = 1$, $b_2 = 2$, $\chi = 0$.

corresponding to (1) in the case of the basin of the uniform depth H_0.

[16] The integration of the last equation is connected with the Hamiltonian system (see e.g. [*Maslov and Fedoiuk,* 1981]); for (15) it is determined by the following 2-D Hamiltonian $\mathcal{H} = C_0|p|$, where $p = (p_1, p_2)$ is a momentum. Thus the corresponding Hamiltonian system has the form

$$\dot{x} = \frac{p}{|p|} C_0, \quad \dot{p} = 0. \tag{16}$$

Consider the one-parametric family of trajectories of this system satisfying the following initial conditions

$$x|_{t=0} = 0, \quad p|_{t=0} = \mathbf{n}(\psi), \tag{17}$$

where

$$\mathbf{n}(\psi) = (\cos\psi, \sin\psi)^{\mathrm{T}} \tag{18}$$

and the angle $\psi \in [0, 2\pi]$ is a parameter. These trajectories are (vector-columns) $X(t, \psi) = (C_0 t \cos\psi, C_0 t \sin\psi)^T$, $P(t, \psi) = \mathbf{n}(\psi)$. The family of trajectories $X(t, \psi), P(t, \psi)$ go out from the point $x = 0$ with a unit momentum $p = \mathbf{n}(\psi)$. The projection of the trajectory $x = X(t, \psi)$ on the plane $\mathbb{R}^2_x$ is called "ray". Here the rays are the straight lines starting from the point $x = 0$ and arriving at the instant t to the point $x_1 = C_0 t \cos\psi, x_2 = C_0 t \sin\psi$. At every instant $t > 0$ the curve formed by the end of the rays is called the "wavefront". Here the wavefronts are the circles $|x| = C_0 t$. One can see from (12) that at the wavefronts the phase $\Phi = 0$. It should be noted also that the points on the front are parameterized by the parameter $\psi \in (0, 2\pi]$ which was introduced in (17) as the initial condition for the impulse $p(t)$. So, ψ in (12) can be considered with the same point of view.

[17] In the case of a basin with a non-uniform depth, asymptotic formulas for $\eta(x, t)$ have a structure similar to that in (12). But the rays are no more straight lines, the Hamiltonian is more complicated, the factor $\sqrt{\frac{l}{|x|}}$ has to be replaced by other one and sometimes the power in the exponent $e^{-i\pi/4}$ has to be multiplied by an integer number m which has a deep topological meaning since it is possible, under certain condition, to evaluate m as the Morse index of the trajectory.

[18] Anyway, it is clear that the algorithm for calculating $\eta(x, t)$ from the (12) is faster than the finite difference method since for finding the value of the amplitude of the tsunami front passing in the point x at time t it is not necessary to integrate over all the space but one need just to identify the trajectory starting from the point where the initial perturbation has taken place and arriving to the point x at time t. In other words this method of integration is "local" in the sense that it is concentrated on the trajectories of an Hamiltonian system. This Hamiltonian system is not difficult to solve numerically in the case of a basin of non-uniform depth.

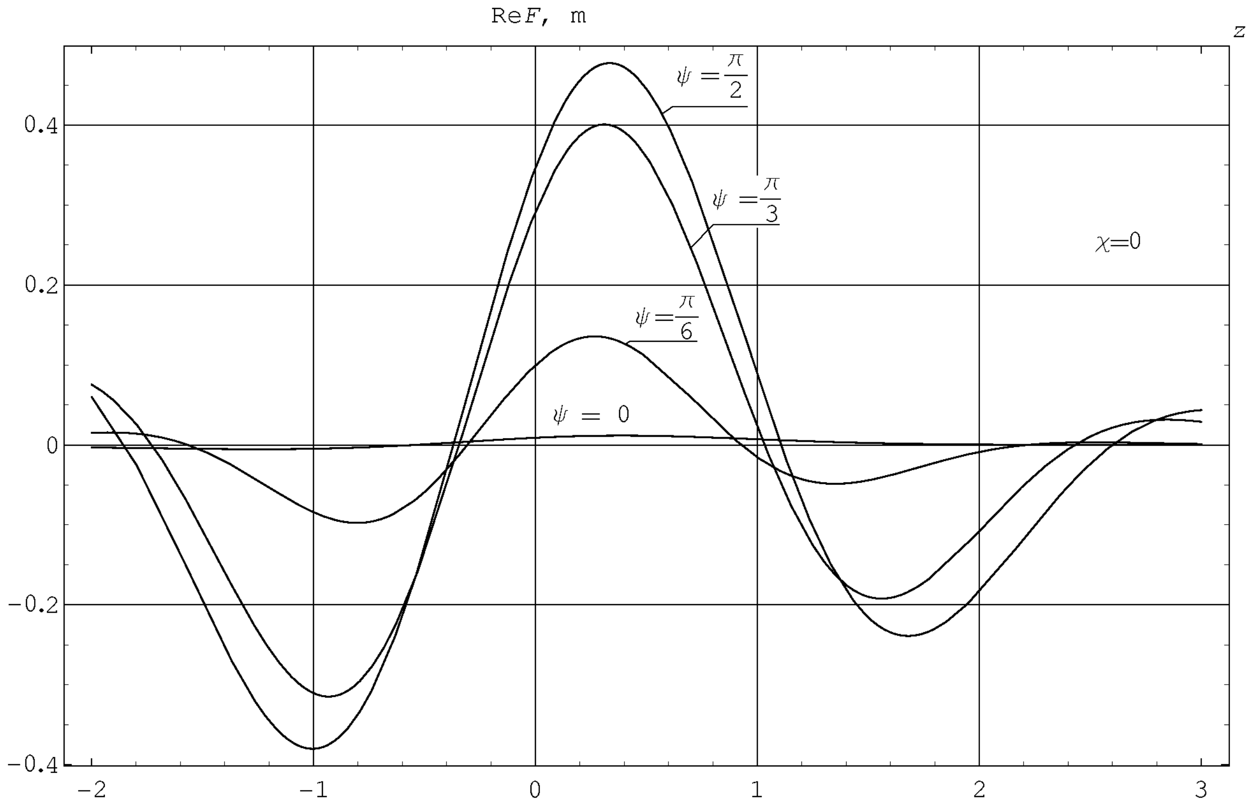

Figure 2. Function $\mathrm{Re}F(z,\psi)$ for $V(y)$ determined by (6) where $V_0 = 1\,\mathrm{m}$, $a_1 = 0, a_2 = 2$, $b_1 = 1$, $b_2 = 2$, $\chi = 0$.

3. Asymptotic Formulas for the Wave Profile in the Non-Uniform Depth Basin

3.1. Relationship Between Fast Oscillating and Localized Solutions

[19] In this section we begin an asymptotic analysis of Cauchy problem (1, 2). We use here well known objects and their characteristics which one can find in books connected with the semiclassical asymptotic and ray method, geometrical optics and wave fronts, Hamiltonian mechanics, catastrophe theory etc. We try to collect here all necessary concepts and give their description in elementary form. A more complete presentation and details can be found in [*Maslov,* 1965; *Maslov and Fedoiuk,* 1981; *Arnold,* 2001; *Babich and Buldyrev,* 1991; *Kravtsov and Orlov,* 1990].

[20] We introduce a parameter

$$\mu = \frac{l}{L} \tag{19}$$

expressing the relationship between the characteristic size of the source l and the characteristic length L of the interval of slow varying depth. Our asymptotic expansions are derived under the assumption that parameter $\mu \ll 1$ and $C(x) = \sqrt{gH(x)}$ is a slowly varying function.

[21] The problem now is to find asymptotic solutions $\eta(x,t)$ to the wave equation with variable coefficient. They can be expressed by means of the wavefront formed by rays (an accurate definition is given in the next subsection). One has to introduce curved rays and characteristics given by 1-D family of trajectories $P(t,\psi), X(t,\psi)$ of an appropriate Hamiltonian system. This Hamiltonian system can be found using a WKB expression for $\eta = A(x,t)\exp(iS(x,t)/h)$ (with some small artificial parameter h) inserting it in the equation and considering the equations of zero and first orders. The first order equation is the Hamilton-Jacobi equation similar to (15) but with coefficient $C(x)$ instead of C_0.

[22] The solutions of the corresponding Hamiltonian system define trajectories which are not straight lines as in the case with the constant coefficient C_0, but are curves. We mentioned before that WKB solutions do not describe the localized solutions. In order to pass from oscillating solutions to localized ones we introduce a new variable ρ, put $h = l/\rho$, multiply WKB solutions by some decaying (as $\rho \to \infty$) function $g(\rho,\psi)$ and integrate this product over ρ from 0 to ∞. We obtain a function localized in the neighborhood of the points $S = 0$ which determine the front. The variable ρ and the angle ψ are similar to that ones appeared in the case of the basin with an uniform depth. The problem is to define the phase $S(x,t)$ and the function $g(\rho,\psi)$ in such a way to obtain the solution of (1, 2). The difficulty is that for $t = 0$ the phase S corresponding to (1, 2) is not a smooth function, and the point $x = 0$ is a (strong) focal point. Thus it is necessary to use the asymptotic representation different from WKB-solutions and we use the Maslov canonical oper-

ator [*Maslov,* 1965; *Maslov and Fedoiuk,* 1981; *Dobrokhotov and Zhevandrov,* 2003]. The realization of these ideas together with boundary layer expansions [*Maslov,* 1973; *Vishik qnd Lusternik,* 1962] near the wave fronts gives not only the phase S and function $g(\rho,\psi) = \sqrt{\rho}\widetilde{V}(\rho,\psi)$ ($\widetilde{V}$ is the same as in (5)), but also global asymptotic solutions to problem (1, 2) which satisfies the initial conditions at $t = 0$ and is correct in the cases with focal and self intersection points on the front etc (see [*Dobrokhotov et al.,* 2006b]). Let us note that to construct the solutions with localized initial data one can try to use the asymptotics of the Green function (see e.g. [*Brekhovskikh and Godin,* 2006; *Kiselev,* 1980]), but during the realization of this approach one has to calculate quite complicated integrals (see [*Dobrokhotov et al.,* 1991]). Our experience show that it is much easier to find explicit formulas applying the ideas mentioned above directly to the problem (1, 2). We present now the final formulas.

3.2. Rays and Wave Fronts

[23] The Hamiltonian system in the case of a basin with non-uniform depth $H(x)$ is:

$$\dot{x} = \tfrac{p}{|p|}C(x), \quad \dot{p} = -|p|\nabla C(x), \qquad C(x) = \sqrt{gH(x)},$$
$$x|_{t=0} = 0, \quad p|_{t=0} = \mathbf{n}(\psi), \tag{20}$$

where $\mathbf{n}(\psi)$ is the unit vector (18). Therefore, the family of trajectories $X(t,\psi), P(t,\psi)$ of (20) starts from the point $x = 0$ with unit momentum $p = \mathbf{n}(\psi)$ (with fixed ψ). Let us indicate $C(0) = C_0$. The Hamiltonian corresponding to (20) is $\mathcal{H} = C(x)|p|$. From the conservation of the Hamiltonian on the trajectories we have the following equation

$$|P|C(x) = C_0, \quad C_0 = C(0). \tag{21}$$

[24] The projections $x = X(\psi,t)$ of the trajectories on the plane $\mathbb{R}^2_x$ are called the "rays". Recall that the "front" in the plane $\mathbb{R}^2_x$ at the time $t > 0$ is the curve $\gamma_t = \{x \in \mathbb{R}^2 | x = X(t,\psi), \psi \in [0,2\pi)\}$, (see e.g. [*Arnold,* 2001; *Maslov,* 1965]). The points on this curve are parameterized by the angle $\psi \in [0,2\pi)$. If $\partial X/\partial\psi \neq 0$ in each point x of the wave front γ_t, then the wave front is a smooth curve. The points where $\partial X/\partial\psi = 0$ are named "focal points". In these points the wave front looses its smoothness. In the situation in which the focal points appear, (they are very interesting from the point of view of tsunami), it is reasonable to introduce the concept of wave front in the phase space $\mathbb{R}^4_{p,x}$ at the moment $t > 0$, i.e. the curve $\Gamma_t = \{p = P(t,\psi), x = X(t,\psi), \psi \in [0,2\pi]\}$. We note that at least one of the component of the vector P_ψ, X_Ψ is different from zero and also the rays $x = X(t,\psi)$ are orthogonal to the wave front γ_t: $\langle \dot{X}, X_\psi\rangle = 0$.

3.3. The Wave Profiles Before Critical Times.

[25] There exist $\delta > 0$ and $t_1 > \delta$ such that a wave front exists but there are no focal points for $t \in [\delta, t_1]$. The first instant of time t_1 at which focal points are formed is called "critical" and denoted t_{cr}. First, we assume that $\delta < t < t_{cr}$. In this case the asymptotic solution is derived in the following way. We define a neighborhood of the wave front for sufficiently small coordinate y, where $|y|$ is the distance between the point x belonging to a neighborhood of the wave front and the wave front. For this aim we will take $y > 0$ for the external subset of the wave front and $y < 0$ and for the internal subset. Then a point x of the neighborhood of the wave front is characterized by two coordinates: $\psi(t,x)$ and $y(t,x)$, where $\psi(t,x)$ is defined by the condition that the vector $y = x - X(t,\psi)$ is orthogonal to the vector tangent to the wave front in the point $X(t,\psi)$, so $\langle y, X_\psi(t,\psi)\rangle = 0$.

[26] The phase is defined by

$$S(x,t) = \langle P(t,\psi(x,t)),\ x - X(t,\psi(x,t))\rangle = \sqrt{\frac{H(0)}{H(X(t,\psi(x,t)))}}\,y, \tag{22}$$

where the second equality is a consequence of the equation (21).

[27] Now we state the first main proposition of this paper.

Proposition 1. *For $t_{cr} > t > \delta > 0$, in some neighborhood of the wave front γ_t, not depending on μ, η, the asymptotic elevation of the free surface, has the form:*

$$\eta(x,t) = \frac{\sqrt{l}}{\sqrt{|X_\psi(t,\psi)|}}\sqrt[4]{\frac{H(0)}{H(X(t,\psi))}} + \mathrm{Re}\left[F\left(\frac{S(x,t)}{l}, \mathbf{n}(\psi)\right)\right]\Big|_{\psi=\psi(x,t)} + O(\mu^{3/2}) \tag{23}$$

Outside this region $\eta = O(\mu^{3/2})$. The function $F(z,\psi)$ is defined in (13).

[28] In order to compute the elevation $\eta(x,t)$ at the point x and time t, one has to find the trajectory of the Hamiltonian system starting from $x = 0$ and arriving at time t in the point x. Then it is possible to compute the phase $S(x,t)$ using the approximation written above. The trajectory is defined (see (20)) by the function $H(x)$ and the angle $\psi(x,t)$ which is the angle between the x_1 axis and the ray arriving at the point x at the instant t from the origin $x = 0$, where the ray was at the instant $t = 0$. So ψ can be find by the solution of equation $x = X(t,\psi)$. The solution exists and it is unique since the vector $\partial X/\partial\psi \neq 0$ before the critical time.

[29] Explicit formula (23) shows that the elevation of the free surface $\eta(x,t)$ is defined by the form of the initial disturbance through the function $F(z,\psi)$ and by the variation of the depth of the basin along the trajectories of the system.

[30] It should be noted that despite of the simple and natural form of (23) its proof is not trivial at all. The main step is to prove the fact that the formula is the same as in the case of constant bottom, if the wave rays are found correctly.

[31] Now we derive some consequences from formula (23). Since the phase $S(x,t)$ is equal to zero on the wave front and $|S(x,t)|/l$ increases rapidly going out from it, then maximum of $|\eta|$ is attained in a neighborhood of the wave front. Moreover, $\eta(x,t)$ can exhibit few oscillations depending on the

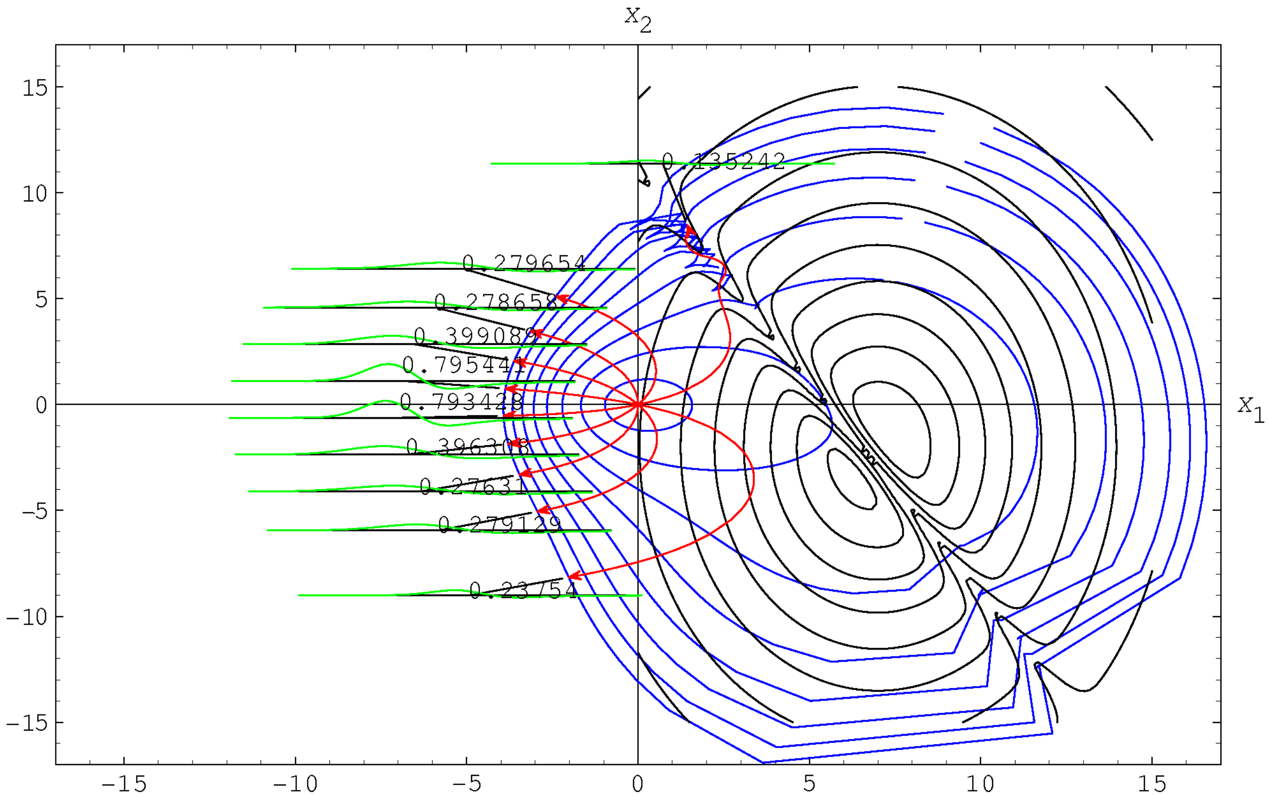

Figure 3. Wave fronts, rays and profiles in the case of the basin with non-uniform depth (see text). $V(y)$ is determined by (6) where $V_0 = 1\,\mathrm{m}$, $a_1 = 0, a_2 = 2$, $b_1 = 1$, $b_2 = 0.25$, $\psi = 0$, $\chi = 0$.

properties of the function $F(z, \psi)$ (which in turn, depend on the form of the initial disturbance, see Figures 1 and 2). The second factor in (23) can be interpreted as two dimensional analogue of the Green law, well known in the theory of tidal waves in the channels: amplitude of η increases as $1/\sqrt[4]{H(x)}$ when the depth $H(x)$ of the basin decreases; the factor $1/\sqrt{|X_\psi|}$ is connected to the divergence of the rays, in other words if a smaller number of rays goes through a neighborhood of the point $X(t, \psi)$, the smaller will be the amplitude of the wave field. The factor $\dfrac{H(0)}{H(X(t,\psi))}$ in the phase $S(x, t)$ (see (22)) expresses the phenomena known as the "contraction" of the wave profile and explains the fact that the wave length of a tsunami decreases when the wave approaches the coast.

[32] We can imagine the following situation. Let two rays start from x =0 with two very different angles ψ_1 and ψ_2, arrive to the wave front in two nearby points due to properties of the function $H(x)$. Let also assume that the values of the function $F(z, \psi)$ are very different for the angles ψ_1, ψ_2 and equal values of z (due to the form of the initial disturbance). Then the amplitudes of $\eta(x, t)$ can be very different at these nearby points.

[33] These effects are illustrated by Figures 3 and 4, where the wave field is pictured by the rays (red lines), wavefronts (blue lines) and wave profiles (green lines). The initial disturbance is shown as a black ellipse located near the origin. The black lines show the contours of $H(x)$ =const, where dimensionless depth is $H(x) = 1 + 3\tanh(2x_1 + x_2 - 11)^2/\cosh^2\sqrt{4(x_1-7)^2 + (x_2+2)^2/25}$ (3-D graph of H(x) see in Figure 5). In these figures the variables x_1 and x_2 are dimensionless and are equal to the same dimensional variables used in the text divided by l measured in km. The wave fronts are shown at the instants $t = 50l, 100l, ..., 400l$ s, where l is measured in km. The wave profiles are shown at neighborhoods of the last wave fronts in the directions of the rays indicated by arrows. The numbers near the wave profiles show its maximum wave height (in m).

3.4. The Wave Profile After Critical Times.

[34] At the instances $t > t_cr$ focal points appear on the wavefront. Now, the elevation $\eta(x, t)$ of the wave in a point x belonging to a neighborhood of this point can be represented as a sum of the contributions coming from different $\psi_j(x, t)$, $y_j(x, t)$, and $S_j(x, t)$ with index j, and with the so-called Maslov index $m_j = m(\psi_j(x, t), t)$.

[35] The Maslov index takes one of the following integer values: 0,1,2,3. It is defined in many ways and containing the topological information of the problem under considerations. We have shown in the paper [*Dobrokhotov et al.*, 2006b] that for the problem (1, 2) one can simplify its calculation connecting $m_j = m(\psi_j(x, t), t)$ with the Morse index

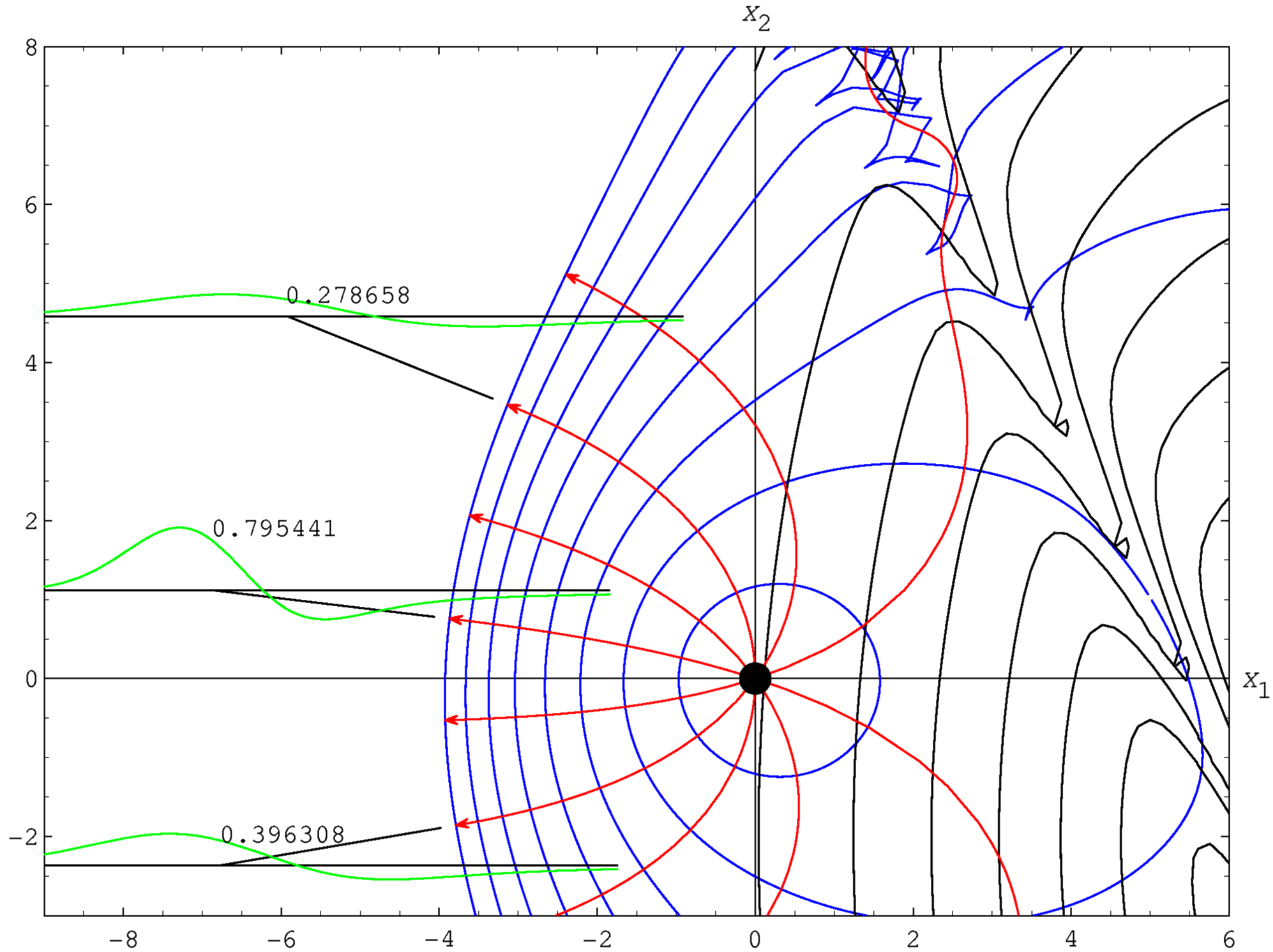

Figure 4. A part of Figure 3.

which counts the number of focal point staying on a trajectory. So this is the Proposition generalizing the formula from Proposition 1:

Proposition 2. *In a neighborhood of the front but outside of some neighborhood of the focal points the wave field is the sum of the fields*

$$\eta(x,t) = \sum_j \{ \frac{\sqrt{l}}{\sqrt{|X_{\psi_j}(t,\psi)|}} \sqrt[4]{\frac{H(0)}{H(X(t,\psi))}} \times$$
$$\mathrm{Re}\,\big[e^{-\frac{i\pi m_j}{2}} F(\frac{S_j(x,t)}{l}, \mathbf{n}(\psi))\big]\}\big|_{\psi=\psi_j(x,t)} + O(\mu^{3/2}). \quad (24)$$

Outside this neighborhood of the front γ_t $\eta(x,t) = O(\mu^{3/2})$. *Again the function* $F(z,\psi)$ *is defined in* (13).

[36] One can see that the indices m_j change the behavior of the functions determining the wave profile (see Figure 6).

[37] Let us emphasize that the number m has a pure topological and geometrical character and can be calculated without any relation with the asymptotic formulas for the wave field. From the Proposition 1, 2 it follows that, in order to construct the wave field at some time t in a point x, one has to know only the initial values $\eta|_{t=0}$ and $\eta_t|_{t=0}$ and has not to know the wave field η for all previous time between 0 and t. The trajectories and the Maslov (Morse) index take into account all metamorphosis of the wave field during the evolution from zero time until time t. In the paper [*Dobrokhotov et al.,* 2006b] some theorems have been shown for connecting those two indices and in the computer program which implements this algorithm there is a simple way for finding the focal points studying the change of the sign of the jacobian of the map. Note also that these formulas are easy to invert for finding the parameters of the shift V from the measures of the wave heights done at some stations.

3.5. Wave Field Asymptotic in a Neighborhood of Focal Point

[38] **3.5.1. Wave front singularities and focal points** To give the complete description of the asymptotic solution to problem (1, 2) one has to describe the asymptotic of the function η in the neighborhood of the focal points. These points are the singular ones on the fronts and one can see them on the Figure 5 on the upper part and on

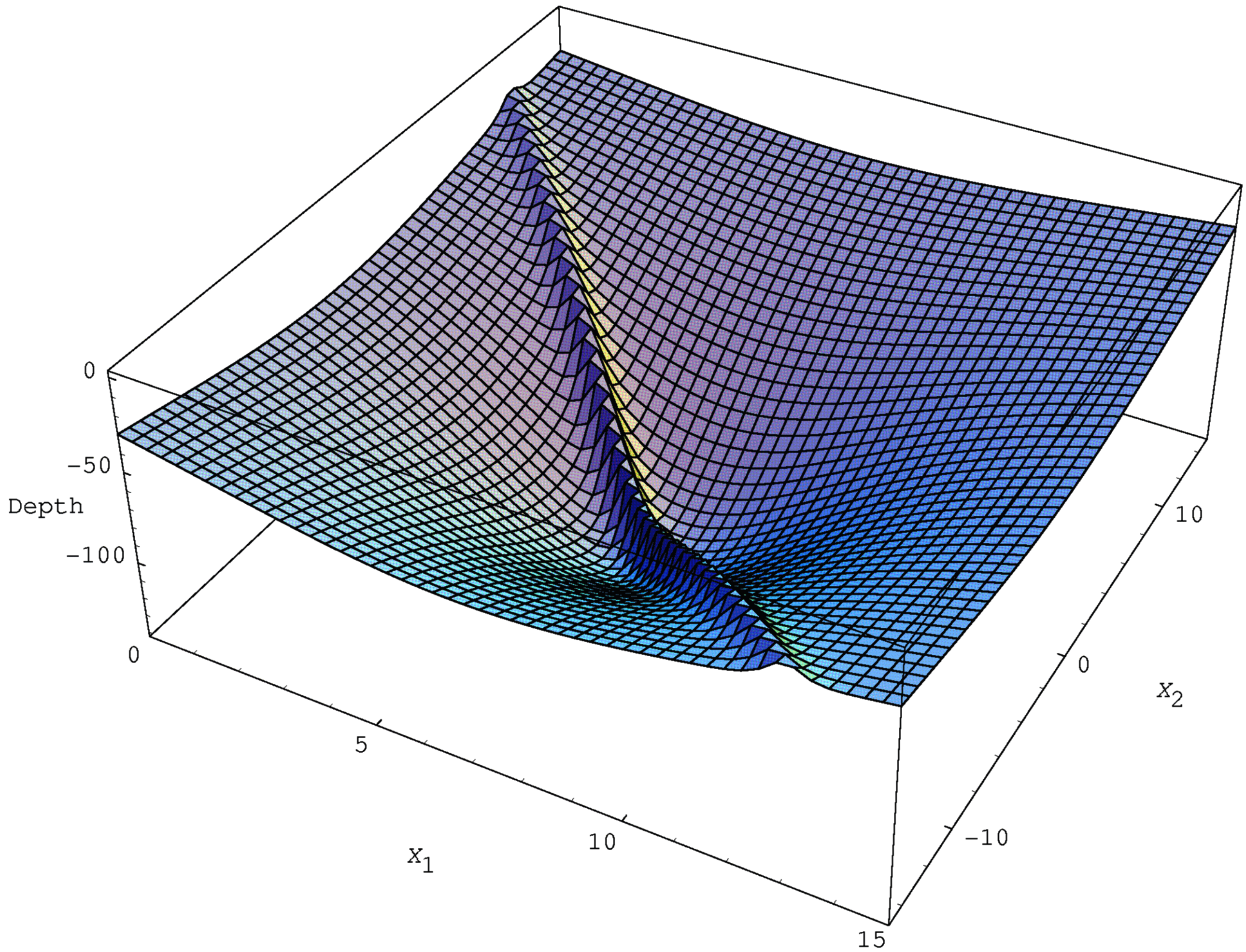

Figure 5. 3-D graph of the function $H(x)$ corresponding to Figures 3 and 4.

the Figure 4 near the right upper corner. They are located over underwater ridge from the Figure 5 and actually connected with the well known trapped waves. The wave field amplitude increases in the neighborhood of these points and depends on the degree of their degeneration. It seams that in real situation only the simplest situation can be realized, nevertheless we give the formulas in a general situation.

[39] **3.5.2. Focal points and coordinate system** So we consider the situation when for some t the point $(P^F, X^F) = (P(t, \psi^F(t)), X(t, \psi^F(t)))$ corresponding to the angle $\psi^F(t)$ is a focal one. In this point X_ψ=0 and one has to use another asymptotic representation for the solution. Roughly speaking the neighborhood of the point $X(t, \psi^F(t))$ on the plane $\mathbb{R}^2_x$ can include several arcs of γ_t with the angles ψ different from $\psi^F(t)$. This means that one has to take into account the contribution of all of these arcs in the final formulas for η in the neighborhood of the point $x = X(t, \psi^F(t))$. The influence of nonsingular points are defined by formula (25) and the influence of the points from the neighborhood of the focal points are described by formulas (31) given below. Thus it is necessary to enumerate the focal points with nearby projections and write $P(t, \psi_j^F(t)), X(t, \psi_j^F(t))$. These points have the same position $X^F = X(t, \psi_j^F(t))$, but different momentum $P^F = P(t, \psi_j^F(t))$. To simplify the notation we discuss here the influence on η of only one focal point omitting the subindex j but keeping P^F.

[40] We present the corresponding formula under the assumption that some derivative

$$X_\psi^{(n)F} = \frac{\partial^n X}{\partial \psi^n}(t, \psi^F(t)) \neq 0, \tag{25}$$

and the derivatives $X_\psi^{(k)F} = 0$ for $1 < k < n$. It means that this focal point is not completely degenerate. For future convenience we introduce the "standard" and "mixed" Jacobian

$$\tilde{J} = \det(\dot{X}, P_\psi)(t, \psi) = \frac{C^2(X)\det(P, P_\psi)}{C_0}(t, \psi) \tag{26}$$

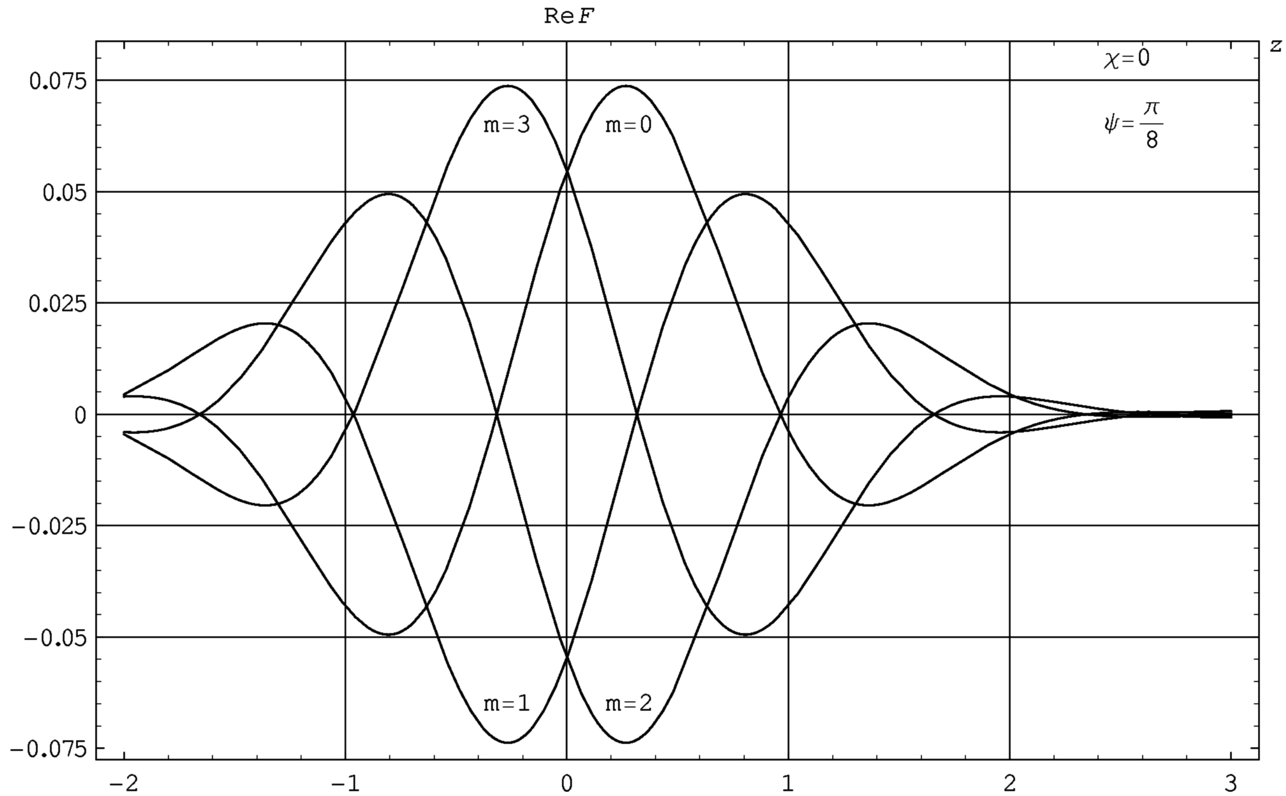

Figure 6. The profile metamorphosis implied by the Maslov (Morse) index: function $\mathrm{Re}[e^{-\frac{i\pi \mathbf{m}_j}{2}} F(z,\psi)]$ for $V(y)$ determined by (6) where $V_0 = 1\,\mathrm{m}$, $a_1 = a_2 = 0$, $b_1 = 1$, $b_2 = 2$, $\chi = 0$ corresponding to the Maslov (Morse) indices $m = 0, 1, 2, 3$.

and some characteristic quantities of the focal point (P^F, X^F):

$$C_F = C(X^F),\ \ \dot{X}^F = \dot{X}(t, \psi^F(t)) = \frac{P^F C_F^2}{C_0},$$
$$P_\psi^F = P_\psi(t, \psi^F(t)),$$
$$\tilde{J}_F = \det(\dot{X}^F, P_\psi^F) = \frac{C_F^2 \det(P, P_\psi)}{C_0},$$
$$J_F^{(n)} = \det(\dot{X}^F, X_\psi^{(n)F}). \tag{27}$$

[41] Again the topological characteristic appears, i.e. the Maslov index of this focal point or its neighborhood (it is the same), but now it depends on the choice of the coordinates in the neighborhood of (P^F, X^F). It is natural to choose the new coordinates (x_1', x_2') associated with the nonzero vector $\dot{X}^F = \dot{X}(t, \psi^F(t))$; namely we assume that the direction of the new vertical axis x_2' coincides with the vector $\dot{X}^F$. We put $\mathbf{k}_2 = (k_{21}, k_{22})^{\mathrm{T}} = \dot{X}^F/|\dot{X}^F| = \dot{X}^F/C_F = P^F C_F/C_0$, $\mathbf{k}_1 = (k_{11}, k_{12})^{\mathrm{T}} = (k_{22}, -k_{21})^{\mathrm{T}}$ and introduce the new coordinates p', x' in the neighborhood of (P^F, X^F) in the phase space $\mathbb{R}^4_{p,x}$ by the formulas:

$$x_1' = \langle \mathbf{k}_1, x - X^F\rangle = -\frac{\det(\dot{X}^F, x - X^F)}{C_F} = -\frac{C_F}{C_0}\det(P^F, x - X^F),$$
$$x_2' = \langle \mathbf{k}_2, x - X^F\rangle = \frac{\langle \dot{X}^F, x - X^F\rangle}{C_F} = \frac{C_F}{C_0}\langle P^F, x - X^F\rangle,$$
$$p_1' = \langle \mathbf{k}_1, p\rangle,\ \ p_2' = \langle \mathbf{k}_2, p\rangle.$$

[42] It is easy to see that

$$\det\begin{pmatrix} \dot{P}_1' & P_{1\psi}' \\ \dot{X}_2' & X_{2\psi}' \end{pmatrix} = \tilde{J}_F. \tag{28}$$

[43] **3.5.3. The Maslov index of a focal point.** The determinant $\tilde{J}_F \neq 0$ in the focal point (P^F, X^F), hence the same inequality takes place in some of its neighborhood, thus $\tilde{J}$ has a constant sign. On the contrary the Jacobian J changes sign in this neighborhood. We define the Maslov index $\mathbf{m}(P^F, X^F)$ of the non (completely) degenerate focal point $(P^F, X^F) = (P, X)(t, \psi^F(t))$ as the index $m(\tilde{P}, \tilde{X})(t, \psi)$ of a regular point $(\tilde{P}, \tilde{X}) = (P, X)(\tilde{t}, \tilde{\psi})$ in the neighborhood of (P^F, X^F) such that the signs of the determinants $J(\tilde{t}, \tilde{\psi})$ and $\tilde{J}(\tilde{t}, \tilde{\psi})$ coincide. For instance one can

choose $\tilde{\psi} = \psi^F(t)$, $\tilde{t} = t \pm \delta$, where δ is small enough. This means that we compare the sign of J with the sign of $\tilde{J}$ on the trajectory (P, X) crossing the curve Γ_t in the focal point (P^F, X^F) before and after this crossing.

[44] **3.5.4. The model functions and the wave profile in a neighborhood of the focal point.** Now we present the formulas for the wave field in the neighborhood of a focal point $x = X^F$. Let us put $\sigma = \mathrm{sign}(\tilde{J}_F J_F^{(n)})$ and introduce the function (or more precisely the linear operator acting on the source function $V(y_1, y_2)$)

$$g_n^\sigma(z_1, z_2, \psi) = \int_{-\infty}^{\infty} d\xi \int_0^\infty \rho d\rho \tilde{V}(\rho \mathbf{n}(\psi)) \times \exp\{i\rho\big(z_2 - \xi z_1 - \sigma \frac{\xi^{n+1}}{(n+1)!}\big)\} = \int_{-\infty}^{\infty} d\xi \int_0^\infty \sqrt{\rho} d\rho \tilde{f}(\rho \mathbf{n}(\psi)) \exp\{i\rho\big(z_2 - \xi z_1 - \sigma \frac{\xi^{n+1}}{(n+1)!}\big)\}.$$

We put

$$z_1^F = \frac{x_1'}{l^{\frac{n}{n+1}}} \frac{\tilde{J}_F}{|\tilde{J}_F J_F^{(n)}|^{\frac{1}{n+1}} C_F^{\frac{n}{n-1}}} \equiv -\frac{\det(P^F, x - X^F)}{C_0 C_F^{\frac{1}{n-1}} l^{\frac{n}{n+1}}} \frac{\tilde{J}_F}{|\tilde{J}_F J_F^{(n)}|^{\frac{1}{n+1}}},$$

$$z_2^F = \frac{x_2'}{l} \frac{C_0}{C_F} \equiv \frac{\langle P^F, x - X^F \rangle}{l}.$$

Proposition 3. *In a neighborhood of the front γ_t each focal point (P^F, X^F) gives the following contribution to the asymptotic values of the solution η*

$$\eta^F(x,t) = l^{\frac{1}{n+1}} \frac{\sqrt{C_0 |\tilde{J}_F|^{\frac{n-1}{n+1}}}}{|J_F^{(n)}|^{\frac{1}{n+1}} C_F} \times \mathrm{Re}[e^{-i\frac{\pi}{2}\mathbf{m}(P^F,X^F)} g_n^\sigma(z_1^F, z_2^F, \psi^F)] + O(\mu). \qquad (29)$$

If several arcs of γ_t belong to the neighborhood of the point x, then one needs to sum over all the corresponding functions (32) *and* (25).

4. Conclusion

[45] In this paper we present the quite explicit formulas for full asymptotical description of solutions of the Cauchy problem with a general localized initial disturbance (source) for the wave equation with slow varying wave velocity. In the case of the Gaussian type disturbance the answer is expressed via the special (hypergeometric) functions. The given description includes: the special trajectories of the simple Hamiltonian system, their first derivatives and the function, implied by the initial disturbance, and the integer numbers (the Maslov or Morse indices) when focal points appear. All objects are well know in geometrical optics and semiclassical approximation, and our main pragmatic result is that only they are needed to construct the final quite explicit formulas for solution to the problem (1, 2) presented in Propositions 1–3. Let us emphasize again that the derivation and proof of these formulas is not simple and use fundamental mathematical theories.

[46] One of the basic conclusions, demonstrated for the source of the Gaussian type (when the answer is expressed via the hypergeometric functions) is that the wave profile crucially depend on the form of initial disturbance of the bottom. As in the real conditions it is very problematic to obtain any detailed information of this disturbance not only at the instant when it happens but and at later times, we propose to develop a more active the researches for application to tsunami warning systems, which used simplified seismic sources. The ones considered in this paper source of the Gaussian type, could be the first ones. We hope also that the given asymptotic formulas can be useful in this application because the visualization of these formulas on a PC is easy and takes not too much time.

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References

Arnold, V. I. (2001), Singulariries of caustics and wavefronts, in *Mathemetics and Its Applications, vol. 62,* p. 276, Kluwer Academic Publishers, Dordrecht.

Babich, V. M., and V. S. Buldyrev (1991), *Short-Wavelength Diffraction Theory,* 456 pp., Springer-Verlag, New-York.

Berry, M. V. (2005), Tsunami asymptotics, *New Journal of Physics, 7*(129), 1.

Borovikov, V. A., and M. Ya. Kelbert (1996), Field near the wavefront for the Cauchy-Poisson problem, *Fluid Dynamics, 31*(4), 173.

Brekhovskikh, L. M., and O. A. Godin (2006), *Acoustics of layered media II, Point source and bounded beams,* 548 pp., Springer-Verlag, New York.

Dobrokhotov, S. Yu., and P. N. Zhevandrov (2003), Asymptotic expansions and the Maslov canonical operator in the linear theory of water waves, I. Main constructions and equations for surface gravity waves, *Russ. J. Math. Phys., 10,* 1.

Dobrokhotov, S. Yu., V. M. Kuzmina, and P. N. Zhevandrov (1993), Asymptotic of the solution of the Cauchy-Poisson problem in a layer of nonconstant thickness, *Math. Notes, 53*(6), 657, doi:10.1007/BF01212605.

Dobrokhotov, S. Yu., V. P. Maslov, P. N. Zhevandrov, and A. I. Shafarevich (1991), Asymptotic fast-decreasing solution of linear, strictly hyperbolic systems with variable coefficients, *Math. Notes, 49*(4), 355.

Dobrokhotov, S., S. Sekerzh-Zenkovich, B. Tirozzi, and T. Tudorovski (2006a), Description of tsunami propagation based on the Maslov canonical operator, *Dokl. Mathematics, 74*(1), 592, doi:10.1134/S1064562406040326.

Dobrokhotov, S., S. Sekerzh-Zenkovich, B. Tirozzi, and T. Tudorovski (2006b), Asymptotic theory of tsunami waves: geometrical aspects and the generalized Maslov representation,

in *Publications of the Research Institute for Mathematical Science, vol. 4,* p. 118, RIMS, Kyoto.
Kiselev, A. P. (1980), Generation of modulated vibrations, *11, Zapiski nauchnyh seminarov LOMI, 104,* 111.
Kowalik, Z., Ed., and T. Murty (1993), *Numerical modeling of ocean dynamics,* 481 pp., World Scientific, Singapore.
Kravtsov, Yu. A., and Yu. I. Orlov (1990), *Geometrical Optics of Inhomogeneous Media,* 312 pp., Springer-Verlag, Berlin.
Lewis, C. L., and W. M. Adams (1983), Development of a tsunami-flooding model having versatile formulation of moving boundary conditions, *The Tsunami Society Monograph Series, 1,* 128.
Maslov, V. P. (1965), *Perturbation Theory and Asymptotic Methods,* 549 pp., Univ. Publ., Moscow.
Maslov, V. P. (1973), *Operational Methods* (in Russian), 559 pp., Mir, Moscow.
Maslov, V. P., and M. V. Fedoriuk (1981), Semi-Classical Approximation in Quantum Mechanics, in *Mathematical Physics and Applied Mathematics 7, Contemporary Mathematics 5,* p. 301, D. Reidel Publishing Co, Dordrecht etc..
Maslov, V. P., and M. V. Fedorjuk (1989), Logarithmic asymptotics of fast decaying solutions to Petrovskii tipe hyperbolic systems, *Math. Notes, 45*(5), 50.
Pelinovski, E. N. (1996), *Hydrodynamics of Tsunami Waves,* 276 pp., IAP RAS, Nizhnii Novgorod.
Shokin, Yu. I., L. B. Chubarov, A. G. Marchuk, and K. V. Simonov (1989), *Numerical Experiment in Tsunami Problem,* 167 pp., Nauka, Novosibirsk.
Tinti, S. (1993), *Tsunami in the world,* 228 pp., Acad. Press, Kluwer.
Vishik, M. I., and L. A. Lusternik (1957), Regular degeneration and boundary layer for linear differential equations with small parameter, *Uspekhi Mat. Nauk,* Engl. translation: American Math. Society Transl., vol. 20, (2), 1962, pp. 239–364 (in Russian), *12*(5), 3.
Whitmore, P. M., and T. J. Sokolowski (1996), Predicting tsunami amplitudes along the North American coast from tsunamis generated in the Northwest Pacific ocean during tsunami warnings, *Science of Tsunami Hazards, 4*(3), 147.

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